MATHEMATICS AND LOGIC

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ABSTRACT

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The aim of this project is to comment on the relationship between mathematics and logic.

Starting with a historical overview of the logicist program, it is found that mathematics cannot be equated to logic. The logicist program is not an appropriate foundation of mathematics. Indeed, mathematics is strong enough without a foundation.

Mathematics is found to have at least one additional humanistic component, art. The idea of the art of mathematics is explored, as both a way of doing mathematics and a way of understanding what mathematics is about. Students do learn to use logic in and out of mathematics and there is an art to doing this. It is recognized that we do not completely understand how or why they aquire these abilities.

Observations of subject's performance on certain activities are used to give us an idea of the actual relation of mathematics and logic.

An introspective analysis of the author's personal experience learning of graduate level mathematics is also offered.

ACKNOWLEDGMENTS

Completion of this project would not have been possible without the help of my family. I am especially grateful to my wife, Flavia and my children Timothy William and Kristy Nicole for their contributions and understanding.

I also thank Vera Dierauer for her help in caring for my energetic offspring and Anna Sierpinska, for whose sure direction and contributions I am very indebted.

I dedicate this work and the completion of my degree to the memory of my late father, Raymond J.P. Corbeil, the person who taught me the importance of learning.

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INTRODUCTION

"The goal of mathematics teaching in the secondary school is to communicate factual knowledge about mathematics to the pupils as well as to develop the necessary skills and abilities for using their knowledge for various practical purposes."

(Talyzina, 1957, p. 51)

In this project, we will explore the relation between logic and mathematics and between logical thinking and mathematical thinking. We will be asking questions such as: Is mathematics reducible to logic? Is it conceivable to have or do mathematics without logic ? This will lead to a broad historical review of logicism and a discussion of the outcome of the logicist program. We will find that logic is an inseparable part of mathematics. We shall propose to think of formal logic, the axioms of mathematics and the art of doing mathematics as forming a basis that spans the space of mathematics.

In the teaching and learning of mathematics the role of logic and the difficulties of logical thinking are not obvious in the activity of proving. We shall reflect, in this project, on what part exactly is logic playing in the activity of proving. The will lead us to discussing the notion of proof ; we shall see that 'proof' encompassed a wide range of activities with very different relationship to logic. The project contains also an account of five (5) interviews with three (2) students and two (2) teachers, meant to uncover, how in individuals of different age, mathematical ability and knowledge, the mathematical and the logical thinking are related to each other.

Chapter I

MATHEMATICAL KNOWLEDGE AND LOGIC

This chapter concerns itself mainly with the idea of **logicism** as a movement in the philosophy of mathematics. Starting with a definition and description of logicism, I will then give a brief historical review of the philosophy of logicism. I then offer a criticism in terms of mathematics and mathematics education, followed by a search for a compromise. This is followed by a discussion of mathematics beyond logic and the “art” of mathematics. Finally, I offer some reflection on the impact of logicism on my thinking.

A Discussion of the Logicist Program

Logicism - “One of the most complicated conceptual labyrinths a human mind ever invented” (Lakatos, 1978, p. 12)

There is no doubt that mathematics and logic share a unique relationship in the field of scientific activity. A biologist studying the relationship might call it symbiotic in the sense that mathematics and logic serve each other and, in a sense, keep each other alive. Maybe the relationship is more than that. Maybe when we look more closely, mathematics and logic are essentially the same animal.

The strict logicist position is that mathematics is simply an “elaboration of the laws of logic." Positing logic as the foundations of mathematics is extremely attractive as it would give mathematics a very strong basis. Unfortunately, some of the results of logicism are simply not tenable. Clearly there exists serious problems with the logicist position and that “today most people regard the logicist program as being at best only partially successful "(Velleman, 1997, p.65).

Main Assumptions Of The Logicist Program

By definition, mathematics and logic are separate sciences. Mathematics is the science of the properties and relations of quantities and logic is the science of reasoning (*New Webster Dictionary*, 1967). Logicism is the “thesis that mathematics or at least some significant portion thereof, is part of logic," I.e. mathematics reduces to logic (Blackburn, 1994, p. 224). Logicism can be further divided into two positions describing the process of reduction of mathematics to logic: expressibility logicism and derivational logicism.

According to ‘traditional’ or ‘strict’ logicism, the truths of mathematics are logical truths, deducible by logical laws from basic logical axioms (Blackburn, 1994, pp. 224-225).

The idea of **expressibility logicism** is that mathematical propositions are purely logical propositions or at least alternate expressions of those propositions. In set theory, for example, propositions of mathematics and propositions of logic tend to be expressed in almost identical ways.

One can also think of logicism as the extraction from, or development of, mathematics using pure logic. The notion that axioms and theorems of mathematics can be derived from pure logic is called **derivational logicism**.

Logicism, then, can be viewed as thinking either that mathematics can be simply derived from logic or that mathematics can be equivalently expressed in pure logic.

Using the certainty and stability borrowed from formal logic, this gives one a foundation for the science of mathematics. Logicism is essentially a result of a program for the search for acceptable foundations of mathematics.

The History Of Logicism: Who and When

Logicism as a philosophy of mathematics has its start among the late 19th century mathematicians and philosophers. Having either exhausted the possibilities in the development of analysis or, more likely, having become increasingly overwhelmed by the ideas being developed and looking for ways to synthesize these ideas, mathematicians began looking at the foundations of mathematics. At the same time a number of mathematician-logicians were working on the foundations of what we now refer to as modern symbolic logic. In addition, later mathematicians would turn to logic as the search for a program for the foundations of mathematics became more significant.

In the late 1800s George Boole (1815-64) made major contributions to logic with *The Mathematical Analysis of Logic* (1847) and with *The Laws of thought* (1854). Boole developed standard form categorical propositions, or Boolean Logic, from the traditional or Aristotelian propositional logic (Boyer, 1989, pp. 647-651; Copi, 1982, p. 192). A direct result is the field of symbolic logic giving a solid language in which one can represent and analyze complicated ideas and arguments, an important tool in the arithmetization of analysis and a basis for later systems of symbolic logic.

Now, as early as 1854 Boole and Augustus De Morgan (1806-1871) argued that logic should be associated with mathematics and not, as the Scottish philosopher Sir William Hamilton (1788-1856)[1] argued, with metaphysics. The advantage of relating mathematics and logic was immediately evident. The roles of logic as both a tool and a language for further abstraction in mathematics are apparent.

> Peacock's algebra of 1830 had suggested that the symbols of objects in algebra need not stand for numbers, and De Morgan argued that interpretations of the symbols for operation also were arbitrary; Boole carried the formalism to its conclusion. No longer was mathematics to be limited to questions of number and continuous magnitude (Boyer, 1989, p. 649).

As the work in analysis progressed, it seems that more and more of mathematics involved some kind of logico-deductive structure. In the year of 1872 Charles Méray (1835-1911), Karl Weierstrass (1815-1897), H.E. Heine (1821-1881) and Georg Cantor (1845-1918) made "crucial contributions to the arithmetization of analysis" involving notions in set theory especially.

> The arithmetization of analysis aimed at showing that all properties of real numbers can be defined with the help of the concept of natural number and elementary set theory (Boyer, 1989, p. 636).

The "Euclid myth" placing Euclidean geometry describing the properties of space at the highest level of existence and reliability in human knowledge propagated the conception of mathematics as maybe a higher logic or merely as an extension of logic (Davis, 1981, p.368). When the discovery of non-Euclidean geometry re-aligned towards foundations based on arithmetic, leading to sets and eventually to set-theory-logic, the imposing position of mathematics as an extension of logic was enlarged.

If Boole planted the seeds, Frege, Russell and Whitehead are certainly the trees. Gottlob Frege (1848-1925) starting with his first work, *Begriffschrift* or *Conceptual Notation* (1879), introduced a new notation for logic "would give it a

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[1] Not to be confused with the Irish mathematician Sir William Rowland Hamilton (1805-1865)

much wider base than Aristotelian logic and could be used to deal with issues in the philosophy of mathematics." (Boyer, 1989, p. 667). As a notation it was not a success, but it signaled a complete break from traditional logic. “Inspired by Leibniz’s claim that arithmetic could be reduced to logic and by his own success with mathematical induction”, Frege began in earnest the logicist program (Boyer, 1989, p. 666). *Die Grundlagen der Arithmetik (The foundations of Arithmetic*, 1884) and *Grundsetze der Aritmetik* (1893) “attempted to show that arithmetic could be derived from the laws or axioms of a system of formal logic," advancing the logicist program (Hamlyn, 1987, p. 289). To accomplish the logicist program, it was therefore enough to prove that the theory of natural numbers would then be developed on the basis of logical concepts only.

Bertrand Russell’s (1872-1970) *The Principles of Mathematics* (1903) and then *Principia Mathematica* (1910-13) written with Alfred North Whitehead represent an even more formal attempt at the same logicist program. A large portion of mathematics was expressed and developed in deep formal logical terminology with seemingly few problems apart from the heavy and deep difficulties involved in understanding and working within the system. Later, some important paradoxes were identified, not to mention the result of Gödel’s work.

In *The Foundations of Mathematics* , an essay written in 1925 by R.P. Ramsey (1978, pp. 152-213), we find the predominant view of the trend in mathematics in and around 1925 (see also Boyer, 1989, pp. 673-693). Ramsey’s account of the foundations of mathematics is based on *Principia Mathematica*, representing the common position of the time. The position is firm and very well developed: mathematics is part of formal logic. Many mathematicians and philosophers truly believed that all that was left were the details and some unimportant loose threads, such as paradoxes in the theory.

In 1900 at the International Congress of Mathematicians in Paris, David Hilbert (1862-1943) gave a talk entitled *Mathematical Problems*. Among the "23 problems designed to serve as examples of the kind of problems whose treatment should lead to a furthering of the discipline" Hilbert asked "whether it can be proved that the axioms of arithmetic are consistent." Thirty-one years later the young mathematician Kurt Gödel proved that within a rigidly logical system, propositions can be formulated that are undecidable or undemonstrable within the axioms of the system (Boyer, 1985, p. 685; Copi, 1979, pp. 333-336). In essence, one of the most important features of the logicist program, i.e. the consistency of the set-theoretic system proposed as the foundation of mathematics, could never be established.

The reaction to Gödel's work was a quiet surprise. The change in thinking required much contemplation and it would take time. The change in view, no doubt, was no less than those having to incorporate a Copernican view in the mist of a "Earth-centric" world view. Even so, one must also recognize another factor for the quiet reaction: the focus on World War II and related pre-occupations. While intensive research in foundations continued in set theory semantics and theory of models, there remained a depression of trend until the late 1950s with a post war renaissance of interest in logicism.

W. V.O. Quine's philosophy is one example of the renewed logicism. Quine's philosophy, sometimes referred to as a "naturalized epistemology", seems to be logicism with a vengeance, as all of natural science, and not just mathematics is to be reduced to some type of set theoretic system.

The whole of physical science is an integrated and unified system, ultimately reducible to physics and, presumably, described and codified by a theory stated in a first-order language [i.e. using the set theoretic language of mathematics] (Cresswell, 1975, p. 110).

Mathematics is assumed to be equivalent to logic and is merely the language of all physical science. The main task of mathematics is to construct proofs of theorems in axiomatic systems, and this activity can be seen as proving logical laws. One possibility is having theorems with mutually contradictory sets of axioms, referred to as 'pluralistic logicism'.

Steven J. Wagner (1992, p. 65) offers a very contemporary logicist viewpoint reclaiming a mostly traditional interpretation of logicism. Rejecting the structure of Pierce, Quine and Wittgenstein's "new epistemology", traditional logicism is defended using a 'rationalist mathematical epistemology'. The hypothesis that "Mathematics is essentially a structure of proofs, which are deductions from axioms," is assembled using an 'a priorist model'. Wagner reformulated Russell and Whitehead's position using a contemporary approach, but essentially leading to some of the same exact difficulties.

Hilary Putnam's (1975, pp. 12-60)version of logicism is a departure from those of Frege, Russell and Whitehead at the turn of the century and is certainly at odds with either Quine or Wagner's contemporary views. Putnam first reformulates Russell's earlier "if-then-ist view" of logicism, essentially rejecting the thesis that mathematics is logic in the strong sense outlined in Russell's

Principia Mathematica (Putnam, 1975, pp. 38-39): “logic is a part of mathematics” but “mathematics is not just logic." The relation between mathematics and logic is reduced from an equivalence to an implication. The logical structure Wagner has taken as ‘mathematics’ Putnam posits as simply a structure within mathematics, and no more.

We can summarize the outcome of the logistic program into three basic categories. First, as a strict naturalized epistemology similar to Quine that reduce mathematics to a formalized system of set theory. Second, as a traditional defense of logicism based on contemporary arguments. Third, as a ‘soft’ logicist argument claiming that logic is simply part of mathematics either in the sense of a sub structure of mathematics or in the purer ‘if-then-ist’ version propounded by Putnam.

Criticism Of Logicism And Search For A Compromise

From the point of view of a philosopher of mathematics

For a philosopher of mathematics, the strict logicist viewpoint lacks relation to the development of mathematics as a science and this I consider one of logicism greatest weaknesses. The logicist program is essentially a program to reduce the science of mathematics to a theory that bears little resemblance to how we use or [often] understand much of mathematics.

If one considers the discovery process of mathematics, what even the strongest proponents of contemporary logicists accept is at least partially incompatible with strict logicism.

Wagner understood that "much of mathematics is not certain in any strong sense" and that "hypothetico-deductive reasoning or (explanatory inference) plays an important role" (1992, p. 68) in mathematics and the structure of mathematics. It is unclear how, then, a traditional logicist might include this aspect of mathematics in their philosophy of mathematics.

Indeed, the reality of mathematical discovery is often intuition based and not the result of a deductive process (Lakatos, 1978, p. 5). Calculus, for example provided valid tools for modern mathematics yet the formal analysis came only centuries later. Some of the most respected mathematicians are responsible for few or no proof.

Galois, for example, postulated his entire theory without explanation or demonstration. It would be very weak to claim that these are either examples of deductive developments of mathematics or to claim these are poor examples of good mathematics.

Following Carl Popper's analysis (1970, p. 57), logicism ignores the impact and connection that the community of mathematicians has with the knowledge of mathematics. Logic is imposed onto the mathematics to give the science certainty and formalism. The formalism and logic are not the mathematics but make for good mathematics, perhaps as part of mathematics. Hatcher (1992, p.) asserts that there exist fundamental points of mathematics, that mathematics is "abstract", and it "consists primarily of reasoning with and contemplating abstractions ... that falsity of a proposition in mathematics is determined by a process of deduction."

A distinction does exist between that which is justified by logic and that which is logic. The error of logicism lies in worrying out the logical structure that is, at most, only part of mathematics and ignoring that there is more to mathematics than the structure. Mathematics is ABOUT "the science" of properties and quantities. Mathematics is not necessarily the properties and quantities in question.

Do we want to buy certainty in the science of mathematics at the cost of understanding? The reduction of mathematics to set theory leaves some basic and fundamental ideas of mathematics as almost imperceptible entities without even achieving the certainty originally sought (Boyer, 1989, p. 251).

Certainly there is some kind of art in working in the science of mathematics that is so much more than logic. There exist certain ideas in mathematics, ideas that are details of properties and quantities in the science of properties and quantities that cannot be reduced to components of logic, and be at the same time completely understood and reasonable, sort of an Heisenberg indeterminacy principle of mathematics (Serway, 1982, pp. 876-877)

Recognizing the logicist dependence on the idea of mathematics as 'a priori' in the sense of being non-empirical in nature can be one key to weakening the logicist position. "Russell tried to establish hierarchy of a priori truths, of 'mathematical beliefs', geometrical or arithmetical" (Lakatos, p. 11, 1978). Logicism without full apriority, states Wagner, is not logicism:

> Any logicist must posit a non-experiential source of warranted belief. ... I am traditional, and logicist, in insulating mathematics from observation and experimentation (Wagner, 1992, p. 69)

The argument that ordinary mathematics is quasi-empirical in nature is interesting (Lakatos,1978, pp. 30-35). The idea that mathematical objects are "merely abstract possibilities" highlights the thesis that logic is simply a structure found in mathematics. "Studying how mathematical objects behave might better be described as studying what structures are abstractly possible and what structures are not abstractly possible" (Putnam, 1985, p. 50; 1975, p. 60).

It turned out that the sophisticated second (and further) generations of logical (or set-theoretic) axioms - devised to avoid the known paradoxes - even if true, were not indubitably true (and not even indubitably consistent), and that the crucial evidence for them was that classical mathematics might be explained - but certainly not proved - by them (Lakatos, 1978, p. 30).

Even the strongest set-theories approaches, like Russell's Principia Mathematica and Quine's New Foundations and Mathematical Logic, have this quasi-empirical element. The set-theorist's defense or explanation of the empirical elements in their set-theories, that empirical data only "compensate for human limits of memory and attention" does not fully account for the nature of understanding. Even if one is able to verse and argue mathematics at a level described as non-empirical, the actual human understanding of mathematics is tied to our understanding of the mathematics at the level of empirical ideas. That we can not remove.

The issue may be that reductionism is not possible in any complete form ... or maybe that there exist posterior components of mathematics. It may be enough to simply point out that we understand, apart or distinct from the issue of belief, some higher mathematics concepts simply based on "touchable" or empirical knowledge and that itself is enough to punch some holes in apriority.

From the point of view of a mathematics educator

One outcome of logicist program is the development of a particular image, one not entirely positive, of mathematics and mathematicians. The image of the mathematician as the coldly logical academic separated from real life is not unusual. This image is personified in the Holywood-esque high school nerd with the pocket protector and slide rule. At least other scientists have the image of the “nutty professor”: a little off but at least occasionally exciting.

Recently I asked, “Why is there the insistence to continue to pursue the logicist program?”. The resulting fields of mathematics are broad and we do not need the so called Hilbert Program to sustain research in this area. Perhaps we should stop to consider what effect the logicist program might have on mathematics education? What do mathematicians want from a protracted search for the foundations of mathematics?

The pedantic search for logical foundations for mathematics has painted the coldly logical image of the bookish, boorish and babbling mathematician in the academic tower. Such images have developed false expectations of what mathematics should be about and what careers in mathematics might offer for students. We have enough difficulty motivating students and perhaps some problems in motivation comes from expectations about mathematics which in turn come from this overtly logical reductionism.

It is unfortunate that some people rarely seen mathematics as an exciting field of discovery, a field of science in all the senses including experimentation.

Compromise

Obviously, mathematics would be poorer without the logicist program. Consider that while set-theory is an example or one positive result of the program, the search for foundations of mathematics and the logicism program resulted in "one of the most complicated conceptual labyrinths a human mind ever invented" (Lakatos, p.12, 1978). Perhaps the continued 'defense' of logicism is just some type of dogmatic logocentrism, an excessive faith in the stability of meanings, or excessive concern with distinctions, or with the validity of inferences, or the careful use of reason (Blackburn, 1994, p. 224). We have somewhat excessive faith that having some foundations of mathematics will result in establishing meaning, truth and stability.

What compromise might be reached concerning logicism? Conceivably, in the same way proof is arguably an essential part of mathematics and at the same time one can do brilliant and imaginative mathematics distinct or far from direct relations to proof, we can have a strong logic-mathematics relationship. In some way pluralistic logicism may be a compromise.

The thesis that the main task of mathematics is to construct proofs of theorems in axiomatic systems, and this activity can be seen as proving logical laws is far too limiting as a description of the activities mathematicians claim to the field of mathematics. An alternate hypothesis could be in the definition of mathematics to include the activity of constructing proofs of theorems in axiomatic systems: one of many important tasks, as opposed to a definition of mathematics solely by this activity alone. Posit logic as either the structure, a tool of, or part of mathematics, but rejecting the reduction of mathematics to logic.

Possibly more interesting is Putnam's suggestion that we abandon the search for foundations entirely. While Putnam salutes and hopes for the continuation in the fields of mathematics resulting from the search for foundations of mathematics, he suggests some difficulties of the contemporary philosophy of mathematics can be resolved by admitting that mathematics is not immune to the same disease as the other sciences: mathematics does not need or want the foundations being searched for.

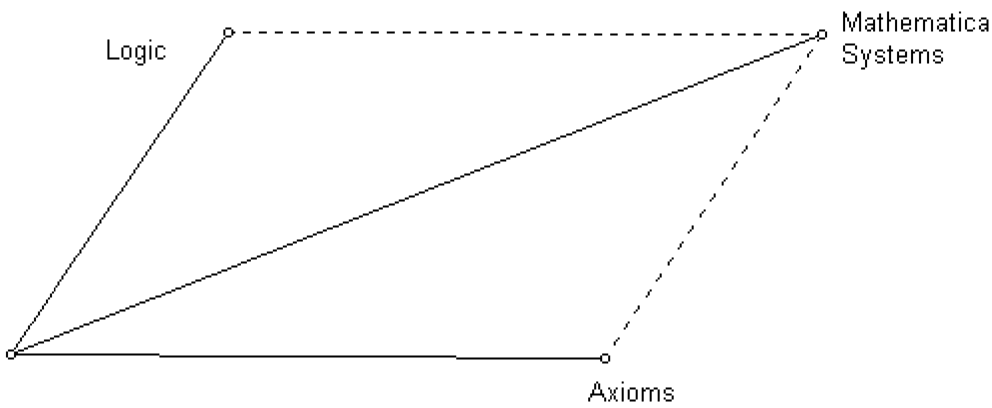
Philosophers and logicians have been so busy trying to provide mathematics with a 'foundation' in the past half-century that only rarely have a few timid voices dared to voice the suggestion that it does not need one. I wish here to urge with some seriousness the view of the timid voices. I don't think mathematics in unclear; I don't think mathematics has a crisis in its 'foundations'; indeed, I do not believe mathematics either has or needs 'foundations'. The much touted problems in the philosophy of mathematics seem to me, without exception, to be problems internal to the thought of various system builders (Putnam, 1975, p. 43).

Is There Mathematics Without (or Beyond) Logic ?

Logic is invincible because in order to combat logic it is necessary to use logic.
(Pierre Boutroux as quoted in Barrow, 1992, p.15)

One question remains: "Can we have mathematics without logic?". We cannot argue about details of logic without at the same time using logic to argue and counter argue (Barrow, 1992, p. 15), we cannot do mathematics without using logic to think about and do the mathematics. Mathematical thinking is analytical thinking. We think and do mathematics using definitions and properties with tools or rules of logic.

Consider Barrow's demonstration of the interplay of the axioms of mathematics and the logic used to manipulate and define using these axioms "spanning" the space of "mathematical systems":



We can envision a mathematical system to be built upon two foundations: ‘axioms’ which are things that are assumed to be true, and ‘logic’ which are the rules by which one is allowed to deduce new truths from the axioms. We can think of the axioms and logic as two sides of a parallelogram and the resulting mathematical system as the resultant diagonal. If one alters either the side marked ‘axioms’ or the side marked ‘logic’, then a new parallelogram is produced with a different resultant diagonal. Moreover, we see that it is possible for the same diagonal to arise from quite different combinations of logic and axioms. (Barrow, 1992, p. 17)

There must be more to mathematics than these two components, logic and axioms. Consider the mathematics of Galois.

Èvariste Galois (1812-1832) was an extremely personable and flamboyant character. Twice rejected as a mediocre student, then accepted for a short time and finally expelled from the Ècole Normale for his hot political viewpoint, the unlucky Galois essentially remained outside mathematical community while alive. The theorems he wrote, without proof or justification the night before he died in a duel over a “coquette”, were as flamboyant as his character and simply incredible for nineteen years old. Such a character makes me think of the stereotype of people is another field of study .... artists !

The Art of Mathematics

A mathematician is one who enjoys mathematics, a connoisseur of mathematics (Tymoczko, 1993, p. 72)

Putnam, Lakatos and many others have suggested that there is some part of mathematics that is not necessarily logic. It could not simply be just the "axioms" for there is more to mathematics and the discovery process within the field of mathematics. We have already suggested that the discovery process is not necessarily deductive in nature.

Tymoczko has made a whole hearted attempt in his argument for grouping and/or comparing mathematics to "fine arts". He asks "Can we treat mathematics as an art?" (Tymoczko, 1993, p. 67).

Aesthetic criticism can be used as the ancillary practice of criticism of mathematics. Mathematicians, as a society or group, chose among the more than 200 000 new theorems produced every year using some sort of criteria (Davis and Hersh, 1981, p.23). Tymoczko suggests that these decisions are essentially based on "mathematical art" aesthetics: "Aesthetic judgments, value judgments in the narrow sense, can [and have] serve[d] as selection criteria for mathematics" (Tymoczko, 1993, p. 68).

Further, one can talk about artists, composers and performers:

> If we can follow this flight of fancy a bit more we can begin to see the mathematics lecture hall, be it a class room or a conference, as it were a concert hall or small parlor where some have gathered to attend a performance. The (performing) mathematician presents a proof, recreates for the audience the lived work of a discovery composed by himself of another. The audience, whether it is faculty or students, we imagine listening for the pure enjoyment of the piece. In presenting the proof, the mathematician is also functioning as a critic, not only in his initial selection of the proof, but in his organization and emphasis (Tymoczko, 1993, p. 72).

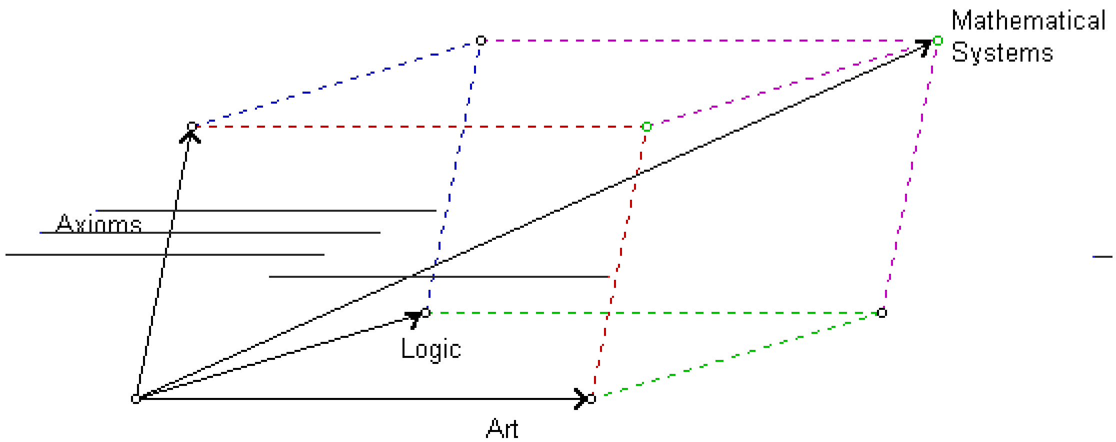
Think about the concept and function of a lemma in a proof. A lemma is a "preliminary or preparatory proposition laid down and demonstrated for the purpose of facilitating more important that follows" (New Webster Dictionary, 1967, p,.487) Why do we need a lemma? We introduce a lemma because in the proof we need an element that is unexpected, remote and non-intuitive to the demonstration. The lemma is not “natural” in the construction of the proof. “Lemmas are a matter of presentation (and composition when the composer is writing with the performance in mind) (Tymoczko, 1993, p. 75).

The idea of proof and of acceptance of a theorem by the mathematical community is not based solely on either the logic nor on the axioms of the proof, but also on other characteristics.

> The performance side of a proof must be capable of persuading mathematicians to accept it just as aesthetic criticism can persuade us to accept a work of art” (Tymoczko, 1993, p. 72)

Putnam’s rejection of a search for a foundations of mathematics and Lakatos’ discomfort with the “deductive-scientific” process of discovery both suggest consideration of a third dimension of mathematics, non-logical, non axiomatic yet extremely creative in essence. The creativity of mathematical certainly does not come from the axioms of the system nor from the logical structure. Occaissional results may come from pure logical analysis but this is not where mathematics is created.

Setting aside the question “Is mathematics art?”, one must bow to the obvious: there is at least some kind of art to mathematics. At the same time, one must be careful not to isolate logic from math. So, let us take the original diagram proposed by Barrow and expand it to include the element of art in or the art of mathematics:



We see by this illustration how a mathematical theory may be constructed from a collection of axioms, with a logical structure that helps us generate justification and understanding of definitions and ideas that flow within the system, and the "art" of mathematics.

We can "tinker" with each arm of the system, by choosing either an alternate set of axioms or shifting the basis of logic, maybe from two valued to three valued logic, resulting in the "diagonal" of the system (Barrow, 1992, p. 18). That is to say, one has redeveloped or created a different mathematical system.

The new arm, "the art of doing mathematics" can also be shifted an in turn may effect in creating new mathematical systems.

The "art" of mathematics is the **creative**, less predictable, perhaps not so logical and very **humanistic** aspect of doing and learning mathematics.   Purely speculative, computer based and/or empirically based discovery might be some examples or shifts in the art of mathematics.

The ever growing controversy of what makes a proof can also be viewed in terms of this third aspect, art. Although a proof is most certainly mostly about logic, the art of proving using alternate methods is being popular. Proofs using computer models and, most recently, models based on biological systems (Fallis, 1996, p. 494), have a non-logical guise.

The idea that there is some “art” is similar to Kant’s intuition yet the flavor of the word art does not attach it to a mere type of perception. Kant and Descartes both posit intuition as purely heuristic once axioms have been chosen (Beth and Piaget, 1966, p.15). As Parsons argues, there is something in mathematics that exists, whose simplest form may be described as intuition, maybe as “ ‘something like a perception’ of mathematical objects”. This lends support to the existence of something that is part of the mathematical systems that is not logic and not axioms.

> The role of intuition is in no way limited to dictating the axioms; it is also intuition and not formal logic, which directs the whole of geometrical reasoning ... [resulting in placing] side by side with *formal or syllogistic reasoning*, a new type of reasoning which will be called *intuitive or constructive reasoning* (Beth and Piaget, 1966, p.-16)

The discovery process of mathematics and what student and mathematicians do is not completely described as mere intuition or constructive reasoning. Part of the art of mathematics is intuition or constructive reasoning but the process is more haphazard and chaotic than intuition or constructive reasoning implies. There is a part of the process of doing mathematics that is purely speculative and creative.

The key here are the words creative and humanistic. Consider many of the “tricks” one uses when doing proofs. When proving a theorem, there often comes a step in thinking and manipulation that is totally unexpected and unrelated, most often preceded by the words “since” or “we know that”.

Think back to the first time you considered a proof that made use of the Triangle Inequality. Its appearance was probably sudden and unexpected. It often requires a total shift of perspective because of its sudden usefulness in mathematics:

With experience and increased aptitude in mathematics one gets a feel for using these "tricks" of the mathematical trade. Certain ideas, concepts and objects become crucial at unexpected places. These tricks are non-intuitive and sometimes counter-intuitive. This is especially true when the proof and mathematics in question is totally separated from its historical context.

In historical context, "Pure mathematics is ultimately humanistic mathematics because it is an intellectual discipline with a humanistic perspective and a history that matters" (Tymoczko, 1993, p.11). Historical context also helps educators identify large cognitive steps and allows students to understand the scope of their progression through mathematics. "It took human being thousands of years to progress to the mathematical level of today's high school students" (Tymoczko, 1993, p. 12)

Commenting on Euclid's proof of the Pythagorean Theorem, Davis and Hersh wrote:

> Now, how does one know where to draw these lines so as to reason with them? It would seem that these lines are accidental or fortuitous. In a sense this is true and constitutes the genius or the trick of the thing. Finding the lines is part of finding a proof, and this may be no easy matter. With experience come insight and skill at finding proper construction lines. One person may be more skillful at it than another. There is no guaranteed way to arrive at a proof. (Davis and Hersh, 1981, p. 166)

There is an artistic quality of inspiration, a knack, for want of a better word. A "knack" is an experience of discovery and creativity to doing a certain bit of mathematics that is not intuition reasoning, logic or axioms (Talyzina, 1957, p.99, Barnard, 1996, p.8).

This may be why it is so difficult to understand what students are thinking, why we have such difficulty drawing up programs in mathematics.

Conclusion

When we talk about concepts in one area of study, say linear algebra, we often end up thinking of the concept possibly as it concerns other areas like geometry or group theory. Relationships across 'areas of study' are a recognized strength of higher mathematics.

Trying to melt or reduce these ideas down to a common trivial-logical basis does not seem either advantageous or worthwhile. A unified theory of nature, like Quine's epistemology suggesting one axiomatic basis to all natural science, should not attempt to relate at the lowest levels of understanding. A unified theory should occur at the highest levels of abstraction. The relationship between mathematics and physics, for example, is at the highest levels.

That there may be a component of an art in mathematics only confirms what many mathematicians have thought and felt for hundreds of years. Mathematicians have always preferred to describe their subject using expressions like "the beauty and wonder of mathematics" instead of "cold logical and sterile ".

We cannot, and should not, say that since we are teaching mathematics we are teaching logic. We should also avoid saying the inverse: that teaching logic is teaching mathematics. A non-axiom and non-logical part of mathematics, a “knack” of doing mathematics, suggests that the goals of teaching students how to think logically need to be tempered with an understanding of the role of logic in the particular mathematics in terms of this “knack”.

Chapter II

LOGICAL THINKING AND MATHEMATICAL THINKING - PHILOSOPHICAL CONSIDERATIONS

In this chapter we shall be reflecting on the relations between the ability to think mathematically and the ability to do mathematics or the ability to do mathematical proof. More precisely, we shall be discussing the following questions:

- Is there a problem with teaching proof writing and logical understanding in mathematics?
- Is it true that doing mathematics is equivalent to doing mathematical proofs? In other words: Are there mathematical activities that do not involve proving?
- Is logical thinking in mathematics involved only in the activity of proving or are there other mathematical activities requiring logical thinking as well?
- Does the ability to think logically entail the ability to
  a) do mathematical proof
  b) do mathematics in general
- Does the ability to think logically entail the ability to do mathematics, or is there more to mathematical thinking than just logical thinking? What would that be ?
- Does the learning of mathematics improve one's logical thinking?

Students' Difficulties With Proof As An Educational Problem

Mathematics education literature is full of lamentations over students' lack of understanding of mathematical proof and inability to do proofs.

> Students frequently have an inadequate appreciation of the underlying structure of proofs and a consequent inability to distinguish a correct proof from a flawed one. It is therefore no wonder that they have considerable difficulty in constructing their own proofs, often not knowing how to start a suitable line of reasoning. (Garnier and Taylor, 1997, p. v111)

On the one hand, many students do not get any training in mathematical proving. This is especially true in North America, where formal geometry has almost disappeared from the high school curriculum. On the other hand, it is claimed that students who had a relatively high level of mathematical training still had problems with proof.

> The results confirm the basic hypothesis of our research: most of the subjects favor supplementary checks of an already proven mathematical statement and its formal proof. This holds not only for students with a low mathematical education but also for students who...were supposed to have a strong background in mathematics (Fischbein and Kedem, 1982, p. 131).

In programs with a high concentration on proof reading and writing we still find poor achievement. Senk's study of 2699 high school geometry students found that "only about 30 percent of students in a [one year] course that teaches proof writing master this objective" (1985, p. 449).

Neither does Tall have encouraging things to say about how students seem to use proof. Tall states that "proof very much meant reproducing a sequence of deductions [memorizing?] to establish an important result. Rarely did it grow from a genuine problem" (1989, p. 28). A proof in the classroom is often "going through a sequence of symbolic manipulations that many students find hard to follow, only to arrive at a result they were prepared to accept without any proof. Why is it necessary to prove something that is known to be true?" (Tall, 1989, p. 28).

Indeed, if the student, as a less experienced mathematician, is comfortable with the truth of a statement, then the proof is not used to demonstrate its truth but is an exercise in demonstration. The value of the proof here cannot be as strong as the proof of an uncertain or an unlikely theorem.

The student does not fully understand how acceptability of a general idea or rule is achieved. "If we are truly to address the notion of mathematical proof in the A-level curriculum [upper high school], we must show the difference between asserting something is true on empirical evidence and proving it true by logical deduction from known facts" (Tall, 1989, p.29).

We need to go further and also determine what details are needed even in a deductive or formal proof. Do we start from primary axioms or are some definitions and propositions acceptable? What about the other forms of proof that may turn out to be as strong as traditional formal proofs (Fallis, 1996, p. 494)?

Tall points out that many students feel that it is enough to list only some occurrences of a rule before jumping to a generalization. The justification given is particular: "tell us the value of n and we will use the computer to verify the result" (1989, p.29).

Mathematical proof "requires clearly formatted definitions and statements, and ... it requires agreed procedures to deduce the truth of one statement from another" (Tall, 1989, p. 30). Because of an emphasis on informal methods of inquiry in schools the problem will only grow. "The power of proof can be emphasized nicely by showing how a general algebraic statement covers a far wider number of cases than a specific numerical calculation" (Tall, 1989, p.31).

Tall concludes that we need "experiences that encourage students to make convincing arguments in meaningful situations" (p.32)

It is evident then, for Tall, that more experience with deductive logic is needed, in addition to paying careful attention to how a teacher offers a "proof" and in what "character the proof is offered" (Tall, 1989, p. 32). Thus, if an unspecified rationalization is given, the teacher should not give the impression that a formal proof is being offered. Conversely, if a formal proof is offered, it should be offered in the full style such a proof warrants.

Fischbein and Kedem suggest that the "ordinary student is intuitively inclined towards an empirical interpretation of the validity of an argumentation" (1982, p.131) It is really interesting that students want more evidence, even after a formal and general proof is given. Obviously something is wrong.

Fischbein and Kedem's paper suggests that students believe that something is intrinsically true or self-evident, thus requiring no verification, or something needs to be proved, thus falling into the category of experimental science.

Students seem to go the empirical route for verification, but why? Is it because they really don't understand what proof confers? What do these students think the purpose of a proof is?

When one does not fully understand or internalize an idea, often one will fall upon the most primitive method available. For instance, people often give up when adding or subtracting, instead they will use their fingers, something that "always works". Interviews might give one an idea of what thinking is going on.

It has long been believed that learning Euclid's geometry trains students in the activity of proving. Assessment studies have, however, undermined this belief very strongly. The blame has sometimes been put on the methodology of teaching.

Eugene Smith, for example, challenged the traditional approach of using Euclidean Geometry to teach proof in his 1959 dissertation, *A Developmental Approach to Teaching the Concept of Proof in Elementary and Secondary High School.* Smith's underlying theme is that proof should be a "central thread" to unite the different mathematics found in the high school curriculum. Traditional methods have placed proof as a separate idea or concept that is dealt with in turn within the curriculum of mathematics. Instead, why not build the curriculum up with proof and deduction as the base and unifying idea of mathematics.

> Although there are still vast areas that remain unexplored in the field, there are enough guideposts to provide promising hypotheses for teaching the concept of proof using mathematics as a vehicle (Smith, 1959, p. 4).

Is it really enough to change the way we teach Euclidean Geometry for students to learn mathematical proving? Perhaps a methodology based on more recent educational theory can be of some help. Unfortunately, education theory might not be developed enough to deal with this question. More importantly, perhaps areas of ideas have yet to even be explored.

If it is true that there is more to mathematics than axioms and logic, if there is some part of mathematics such as the “knack” of doing mathematics, then we must come to some better understanding of this feature before we can fully comment on the suitability of Euclidean Geometry, or any other approach, as a means to improving the learning of mathematical proving.

Senk’s work implies that what is needed is not better methodology of teaching but better knowledge about how students learn to prove: "only about 30 percent of students in courses that teach proof writing master this objective" (Senk, 1985, p.454).

Senk concluded that "research on mathematics education needs to identify cognitive and affective prerequisites for doing proofs and techniques for helping students acquire these prerequisites.

The place of proving in mathematical activity

A mathematical proof is a correct and convincing mathematical argument. Whether an argument is correct is a matter of logic: the conclusion must be a logical consequence of the premises (Slomson, 1996, p.11)

In mathematical practice, in the real life of living mathematicians, proof is convincing argument, as judged by qualified judges (Hersh, 1993, p. 389)

What is all this fuss about proof in recent research in mathematics education? In this section we will explore the idea of equating ‘doing mathematics’ with ‘doing mathematics proof’.

Assume that we can equate doing mathematics with doing mathematical proof. Such is certainly the opinion of some: “Mathematics students or graduates who do not know how to construct proofs or how to read them critically have at best an incomplete understanding of any mathematics they are studying” (Reisel, p. 490)

If you are applying a ready-made formula to calculate the area of a cylindrical surface, are you doing mathematics? Although the activity of applying a ready-made formula can be logical and may even resemble a proof in some cases, this activity is not proof, and then, not mathematics.

Is applied mathematics really mathematics? It has been said that applied mathematics and statistics, throughout the history of mathematics is not really mathematics but a lesser or different science. Pure mathematics is certainly closer to the ideal mathematics.

This argument is flawed. Applied mathematics is real mathematics maybe more so than pure mathematics:

> As a mathematical discipline travels far form its empirical sources, or still more, if it is second and third generations only indirectly inspired by ideas coming from "reality", it is beset with very grave dangers...there is a great danger that the subject will separate into a multitude of insignificant branches ... (Borel, 1983, p. 12).

Hanna recently has argued that "we can impart to students a greater understanding of proof and of a mathematical topic by concentrating our attention on the communication of meaning rather than on formal derivation" , suggesting a very strong connection between doing proofs and doing mathematics (Hanna, 1995, p. 42).

Mathematics is not simply the creation and study of formal systems (Tymoczko, 1993, p. 68). Concentration on either formal "derivations" or on "communications" is possibly misguided since neither is what mathematics "is about". A mathematical theorem-proof, considered as a "great work", "has a richness and complexity that a sensitive audience can be brought to acknowledge, and it is the function of criticism to bring them to this stage" (Tymoczko, 1993, p. 74).

Recently, I have had the experience in doing a number of proofs in analysis. Proving was not simply communication. Proofs were sought even after I was completely convinced of a theorem. Finding a more rigorous justification for the theorem was not necessary for me to completely accept the theorem. I was convinced.

Doing and studying the proofs gave me insight into the theorem and related mathematics. I learned much more than merely the theorem's result. I got something out of the proofs that increased my understanding of the mathematics and gave me a taste for looking deeper and enjoying just doing the mathematics.

The ultimate example of this came at exam time. I was intrigued and worked at length on some relevant proofs, including the proof that "An uncountable set of real numbers has a limit point". The exam question was the following:

Let T be an uncountable set of real numbers. Prove that there is an infinite subset $A \subset T$ which is bounded.

At first glance these propositions seem unrelated. But the proofs follow a remarkable similarity in thinking (see Appendix B). The second proof is not a copy of the first, nor do the theorems imply the same result. Proving was not the doing of mathematics here. Doing the mathematics was making the multitude of minute connections between the two theorems and developing a proof of the second theorem.

Often in mathematics, more than one proof can be readily accepted for the same theorem. As Dhombres (1993, p. 401) pointed out, more than one proof "imply[s] the existence of different paths to mathematical knowledge." A second proof can even serve as a “strategic redundancy”, a practice well demonstrated in Gregory St. Vincent’s *Opus geometricum* (1647).

Siu’s review of proof and pedagogy in ancient China stresses a truth found throughout the history of mathematics:

> If the only role of a proof were verification, nothing would be gained by giving different proofs of the same theorem. But different proofs serve not merely to convince but also to enlighten (Siu, 1993, p. 345)

A more modern example comes from the existence of multiple proofs for the triangle inequality. Proofs case by case, usually for more elementary students, and proofs in algebraic manipulation using methods and skills from analysis differ in the audience to which the proof is performed. These skills and methods are as important as the proposition being demonstrated. Each proof gives a different viewpoint of the ideas related to the triangle inequality and makes a connection to a different part of mathematics.

Multiple proofs for the same proposition also weakens the idea that “doing proofs is equivalent to doing mathematics." Otherwise “this contradicts the implicit assumption that there exists only one natural way, only one authorized sequence of logical steps leading to a correct conclusion” (Dhombres, 1993, p.402). This assumption comes from thinking of doing proofs as equated to doing mathematics. There ought not be two distinct objects from one piece of mathematics.

Hanna claimed that proof is at least partly communication. Let us extend this discussion to a reflection of language acquisition and grammar.

Consider an analogy between proving in mathematics and grammar in language writing. Is “doing grammar” writing language? Perhaps some teachers might say yes. The value of teaching grammar using “grammar exercises” is certainly valued in the traditional setting.

Suggesting not to do grammar exercises is often misunderstood and there has been much resistance from the community of educators of language. Obviously we cannot get along without grammar and there is a need to improve grammar over time. Nonetheless, the argument against teaching grammar and using grammar exercises is fascinating and has a good deal to contribute to the ideas of proving as an activity in mathematics.

Grammar plays a role in language much like the role proof plays in mathematics. Language is not reducible to grammar, nor can one have language without some degree of grammar. Sounds familiar?

Perhaps, then, teaching proof is not going to have the result educators have come to expect. Perhaps, like grammar in language, proving and proving skills are acquired by doing mathematics and not by practicing proving. We need to create situations and exercises that allow students to express themselves mathematically and express the truth of mathematical proposition in their own way.

Can we teach logic successfully?

Assuming that it is important to teach students mathematical proof and that logic is important in proving, can we successfully teach students mathematical logic?

Unfortunately, the prospects are not very optimistic. Cheng et al. (1986) found that university students who have taken a course in mathematical logic were not performing any better on logical tests after the course than they had before the course. There are several reasons for this state of affairs.

First, as a warning, one must be wary of basing educational objectives on arguments from what mathematicians do or how mathematicians think. For instance, Hanna also states that:

> The primary implication for curriculum planning...which aims to reflect the real role of rigorous proof in theory and practice of mathematics must present rigorous proof as an indispensable tool of mathematics rather than as the very core of that science ... (Hanna, 1983, p. 89)

Here Hanna commits what Brown calls the naturalistic fallacy (Brown, 1993, p. 109). Hanna describes and argues for what “is” the case in the society of mathematicians. Proof and logic are paramount features for the mathematician. From this the conclusion is drawn, that we “ought” or “must” teach this or that way. One cannot conclude “must” from an “is” argument.

The evidence of using a curriculum based on these “musts” and “oughts” reflects the conclusion that teaching rigorous and formal proving is as successful as teaching grammar.

Hanna suggests that even for a mathematician the rigorous proof mean less than understanding what the theorem says, i.e. what and how it is understood by the mathematician in terms of the mathematics known to that mathematician (Hanna, 1995, p. 42).

Even if a consensus is reached on the place of proof in mathematical activity, this only gives one an idea of a pedagogical approach or methodology, but does not represent a definitive argument for a pedagogy.

Another reason for this state of affairs might be simply normal errors in human reasoning or simply a lack of ability to perform higher abstractions, i.e.

the normal fallibility within human development. We have trouble teaching logic because humans do not use logic well.

Piaget's theory of cognitive development is split in two four basic stages (Wade and Tarvis, 1987, p. 473). The last stage, the formal operations stage (age 12 to adulthood), marks the being of abstract reasoning and adult cognitive development. At this level of cognition, development relies mainly on education, accumulated knowledge, and the integration of experience and observation known as wisdom" (Wade and Tarvis, 1987, p. 509).

The theory suggests that most adults will eventually become capable of this level of cognition, but not all. "Recent research, however, shows that not all adolescents develop the ability for formal operational thought or complex moral reasoning" (Wade and Tarvis, 1987, p. 509).   Although the rationale is not clear, this certainly indicates that some, perhaps many students will be either incapable of, or very poor logical thinkers.   Thus, it is possible that some students cannot be successfully taught logic thinking because of either an incomplete cognitive development or these students are not capable of this level of development (Rathus, 1984, p. 322).

Of course there is a danger in blaming a student's development on their poor learning.   Considering the weakness of research in the psychology of logical thinking, and the fact that other psychologists have rejected Piaget's ideas or at least Piaget's formal cognitive stage, is a strong indication of the need to verify and continue research.

Girotto suggested that the major problem in teaching logic is that there exists one or more cognitive obstacles in the learning of logic (1989, p. 195). When large numbers of students tend to stumble in learning at roughly the same place or level, it is possible that instead of being a developmental limitation there exists a cognitive step of mental accommodation or adaptation that is too large or too high. A cognitive obstacle might develop not from limitations of the learner, but out of the methodology or approach of the curriculum.

The reported success rates by Girotto, Senk, Fischbein, Ainely, Arsac, Talyzina, Bell, Cheng, Ballacheff and many others definitely suggest a cognitive obstacle(s) of considerable magnitude and/or multitude.

Both Girotto (1989, p. 202), with his bee activity, and Boero (1996a, 1996b), with the sun shadows teaching experiment, have reported some success in identifying and then overcoming some cognitive obstacles.

Cheng proposed the idea that humans use a “pragmatic method of reasoning” (Cheng et al., 1986, p. 294). Meaning that rules of reasoning that are in one's schema are based on practical experience using these rules in common, mostly informal, situations.

In some cases these rules correspond to formal rules of inference in logic. In many cases the rules affect or incorporate a single situation. For example, logical implication between two propositions can have four cases. A similar pragmatic rule, like “you must be 18 years old to legally buy and consume beer” considers only the situation of either satisfying the requirement of “18 years old” or not satisfying the requirement.

The same might be true of a mathematical definition. In many situations, one is only interested in whether the situation or object satisfies the definition. How often do we give a definition and then look for many cases where the antecedent is not satisfied?

Why then should we be surprised when we try to now teach and test students in situations that are in almost all practical instances?

These suggestions are not inconsistent from Boomer's opinions on improving language skills.

> But what do we know from theory about the teaching of formal grammar and exercise work? I get sick of saying this. 50 years of research shows that teaching formal grammar is of no use whatsoever in improving writing. You might as well play chess or scrabble. And yet we still get many, many people in their troglodyte caves all around the countryside believing in the old tales and myths and rituals about grammar. Why can't we settle this bogey once and for all? By grammar, I mean the teaching of verbs, adjectives and nouns, filling in the blanks, or putting commas and semicolons in a trial paragraph. We also find a continuing stream of tests which will show that kids can't do things. Have we ever looked to see what kids can do ? (Boomer, 1989, p. 21)

If the analogy between proving in mathematics and writing in language holds, practicing proving by formal work and exercise may not have the outcome we expect. If grammar exercises are exceptionally unsuccessful, then we should expect proof exercises to be fruitless. Certainly our lack of success at teaching proof and proof writing, shown study after study, demonstrates what our students cannot do.

So, since there is a demonstrated use and "natural" ability in students for using "pragmatic thinking", Cheng (1986) and Girotto (1989) advise replacing (with children) or complimenting (with university students) the teaching of formal logic by what they call 'pragmatic training'.

Employing methods and activities that support pragmatic thinking, and offering an interpretation or form that is closer to "pragmatic" thinking, gives better results (Girotto, 1989, p. 201).

As Boomer suggests, let us see what students can do. "We learn language by using language in situations where we intend either to make meaning or to get meaning for good reason" (Boomer, 1993, p. 22). Perhaps students learn mathematics by doing mathematics, i.e. proving should be seen as a student's expression of his/her conception of mathematics.

We can teach logic successfully. Obviously, some students learn. It is, perhaps, only a matter of taking subjects at different levels of ability in logic and mathematical proving, studying the progression of these subjects and identifying the situations that improve understanding and learning.

What is the place of logical thinking in mathematics?

In this section we shall be dealing with the question: Is logical thinking in mathematics involved only in the activity of proving or are there other mathematical activities requiring logical thinking as well?

Proving seems to be only one of many activities in mathematics requiring logical thinking. Compare the following questions:

Evaluate $\int_{-1}^{+1} \frac{1}{1+x^2} dx$

Prove that $\int_{-1}^{+1} \frac{1}{1+x^2} dx = \frac{\pi}{2}$

One question is a calculation and the other is a proof. But, as these questions demonstrate, it is easy to see how closely related a calculation is to a proof. "For a calculation to be seen to be a proof that the answer is correct, its needs to be set out clearly, with adequate explanations" (Slomson, 1996, p.11). Thus, doing calculations can differ very little from doing proofs. On the other hand, even doing calculations, especially more complicated calculations, require adequate abilities in logical thinking.

Similarly, whether you are applying and/or understanding a formal definition, making connections between theorems, or applying a theorem (the modus ponens rule of reasoning) you are using logic in mathematics without proving. One uses logic thinking every time one attempts to understand a mathematical statement or performs a calculation.

Consider the following definition of uniform continuity:

Definition: Uniform Continuity

Let $A \subseteq \Re$ and let $f: A \to \Re$. We say that f is **uniformly continuous** on A if for every $\varepsilon > 0$ there is a $\delta > 0$ (that is dependent on ε) such that if $x, y \in A$ are any numbers satisfying $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Apart from the nested "if-then's of the definition, note the importance of the order of the logical quantifiers in the definition. For every $\varepsilon > 0$, there is exists an $\delta > 0$, or $\forall \varepsilon > 0, \exists \delta > 0$ is not the same as $\exists \delta > 0, \forall \varepsilon > 0$, i.e. for some $\delta > 0$, all $\varepsilon > 0$. A deep understanding of the logical quantifiers is necessary for understanding but no "proving" ability is needed.

To understand a new mathematical statement of even minor complexity, one needs to take it apart piece by piece considering the ordering of logical quantifiers, moods, and forms. Seldin and Seldin (1996. p. 127) call this the "Unpacking the Logic of Mathematical Statement". At the very least, the " 'if-then' pattern in the statement of a theorem [or any other mathematical statement] corresponds to the 'suppose-therefore' pattern in a proof framework" (Seldin and Seldin, 1996, p. 131).

Relations between the ability to think logically and the ability to do mathematical proof and do mathematics in general.

In this section we shall be dealing with the question: What is the relationship between abilities in logical thinking and abilities to do mathematical proofs and doing mathematics in general.

If our access to new concepts or theory in mathematics is only by reading and understanding definitions and theorems, then it is through logical analysis of these definitions or theorems that one comes to understand the concept in question. The non-inventor of a mathematical concept or object must do some kind of logical unpacking in order to come to grips with the mathematics.

There is wide consensus among educators and mathematicians, deliberate or unconscious, pushing for some improvement of the ability to understand and perform logic on the part of the student of mathematics. Understanding and doing appropriate logical tasks are a requisite to acquiring understanding the knowledge base of mathematics, which, at the least, includes some logical mathematics. Doing mathematics does not necessarily mean that one is proving, but it certainly involves logical thinking.

Seldin and Seldin indicate a relation between "logic unpacking" and proving. "Students who can reliably validate proofs also can reliably unpack the logical structure of informally stated theorems" and "students whose unpacking abilities are deficient will also be unable to validate proofs" (Seldin and Seldin, 1996, p. 131). This suggests that if a student can validate a proof, then this student can unpack the logical structure of informally stated theorems.

Consider the theorem L'Hopital's Rule for evaluating limits. Evaluating $\lim_{x\to\infty}\frac{\ln x}{x}$. yields the indeterminate form $\lim_{x\to\infty}\frac{\ln x}{x}=\frac{\infty}{\infty}$. Using the theorem is straightforward but does demand some logical thinking:

$$\lim_{x\to\infty}\frac{\ln x}{x}=\lim_{x\to\infty}\frac{\frac{d}{dx}[\ln x]}{\frac{d}{dx}[x]}=\lim_{x\to\infty}\frac{1}{x}=0.$$

The proof of the theorem is at quite a different level (see Appendix B), requiring some depth in mathematical analysis. It this case, one level of logical ability is needed to understand a theorem at a depth for applying the theorem, and another, higher, level of logical ability to understand the proof of the theorem.

Conclusion

Our reflection on the relations between the ability to think mathematically and the ability to do mathematics or the ability to do mathematical proof is complete. The results of these considerations have uncovered a number of ideas.

There is much evidence that students have problems with logic and proof. In addition, there is evidence that traditional methods of teaching proof are not very successful. This suggests a the need to identify the cognitive and affective prerequisites for doing proofs and a need to identify techniques for helping students acquire these prerequisites.

The identification of cognitive obstacles has proved useful. Girotto and Boero have both conceived of some methods to improve success with teaching skills need for better understanding. Perhaps, teaching proving is not the answer. Perhaps the methodologies need to be related to students doing mathematics as an activity. Doing mathematics does have different levels of logical activities, and educators can use this to improve the students understanding.

Chapter III

Mathematical Proving and Logical Thinking - Results of An Observation

In the previous chapter some conjectures were formulated as to the relations between logical thinking and mathematical proving. In the present chapter, these conjectures will be confronted with the results of an observation of five subjects who were given a certain number of logical and mathematical tasks to analyze or to do. The observation was a 'participant observation': the observer was present during the solution of the problem and was participating in the solution by asking questions.

The Subjects

The observation took place over a period of two weeks at an English public high school in the sector of Pointe-St. Charles, Montreal. The school is noted for relatively low academic achievement and a majority of students are from economically disadvantaged families. Two subjects were students aged 16 to 17 in their eleventh year of study of low to average mathematical ability. The third subject was a science teacher (including mathematics) with three years of teaching experience. The remaining subject was a student teacher (mathematics specialty) in her last stage of work-study. These subjects had no identified learning disabilities. One student subject was observed a second time to follow up on his retention of understanding.

The Tasks

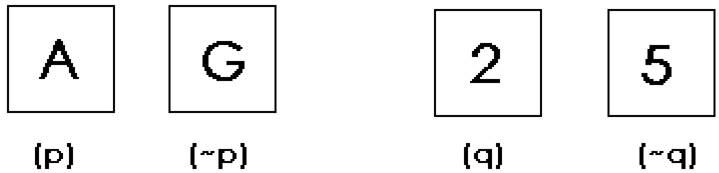
Available to the observer was a bank of 14 tasks. Observations started with Task 1. Depending on the subjects' individual response, the observer selected what he felt was the next task appropriate to the demonstrated level and thinking of the subject, and so no two subjects had the same set of tasks. The general purpose was to get a general feeling for the subjects' ability and level of logical maturity with mathematical or non mathematical content.

After each observation, subjects were shown the correct solutions to the task they had attempted. The Johnson's Envelopes Task was used as a teaching exercise with one subject, who was then observed two weeks later and asked to redo the Wason's Card Task. The second observation was intended to asses this subject's retention of understanding after a two week period, i.e. 'Was the Johnson's Envelope task useful as a teaching tool?'.

Task 1: Wason's Cards

This was an object based task based on the work of the British psychologist Peter Wason in the late nineteen sixties (Girotto, 1989, p. 196). The subject was exhibited four cards laying flat on a desk top. Each card had a letter on one side and a number on the other. Two numbers and two letters are showing face up.

Figure 1: Wason's Cards



The task was explained to the subject by generally stating the following:

> Each card has a letter on one side and a number on the other. Consider the following statement: "If there is a vowel on one side of the card, then there is an even number on the other side" [observer writes this on the blackboard]. I would like you to verify if this statement with reference to these four cards here [pointing to four cards on desk]. Take your time and think about it first. Then, select cards you would want to flip over in order to verify this statement. Try to choose the minimum or least number of cards possible to verify the statement. You might need to choose only one card, or up to four cards. Do you understand what is expected ?

The observer made sure the subject understood what was expected. The subject would make his/her choices. The observer would simply ask the subject for the justification for the choice of selection using open ended questions like "Why did you chose these two"?

Solution

Choosing to verify or turn over the cases of p and ~q, i.e. the vowel and the odd number.

Purpose

The if p then q initial base statement is a conditional statement, also called a material implication. The purpose of this task was to identify how well the subject could "control" the logical rule of implication for the characteristics of "vowelness" and "eveness". Working with the rule requires thinking about the four possible cases of a material implication. This is illustrated by the following truth table:

Case	p	q	p --> q	~ p	~ q
1	T	T	T	F	F
2	T	F	F	F	T
3	F	T	T	T	F
4	F	F	T	T	T

Note that only case 2 is of concern for verification, and so the correct solution would be choosing to verify or turn over the cases of p and ~q. Meaning, one is checking the cards with a vowel and an odd number facing up.

A facing vowel (p) represents either case 1 or 2, meaning either the whole implication is true or false. A facing non vowel is case 3 or 4, neither requiring checking to verify the whole implication. A facing even number would be case 1 or 3, also not of concern.

The case of an odd number or a non even number (the assumption here is that all numbers given are either even or odd) is relevant since case 2 is again in contention. We need the information on the other side of the card to verify if this a case 2 or case 4. This is the most difficult aspect of this task.

This is a very formal account of the solution and follows my own thinking which is influenced by knowing the solution and purpose beforehand.

An alternate account might be checking if the implication “vowel->even” is false rather than to try to find confirming evidence for its truth. Implication is false only is one case; it is true in three cases, so checking if the implication “vowel->even” is false is more sensible.

The implication would be false in the case of the premise (there is a vowel) and a false conclusion (there is an odd number on the other side). So if I look under A (premise true) and discover an odd number, the statement is refuted and I am done.

If I find an even number under A, I am not done yet, because I do not know if the statement holds in all cases or only in this one. Thus, is this case there is an even number under A. I have to take a second step.

It does not make sense to look under G, because this card does not satisfy the premise of the statement we are checking. The question is thus what is under 2 or 5.

Checking under 2 would not be conclusive, because, if I found a vowel, I would have another confirming evidence, but not a proof. If, on the other hand, I found a consonant, the case would not satisfy the premise of the statement and would be useless in its verification.

It remains to look under 5. This is a good case, since, if I find a vowel under it, the statement is refuted. If I find a consonant, it is proved. So, the statement can be checked in, at most, two steps.

Expectations

All subjects were expected to choose the facing vowel (p) as this case to more concrete and real world experiences. It would not have be surprising to have all student subjects fail to correctly identify the second card, the ~p case, as the reported success rate is about 10% (Girotto, 1989, p. 197; Senk, 1985, p. 448, Fischbein, 1982, p. 133) . The adult teachers were also expected to have at least some minor difficulty identifying the ~p case.

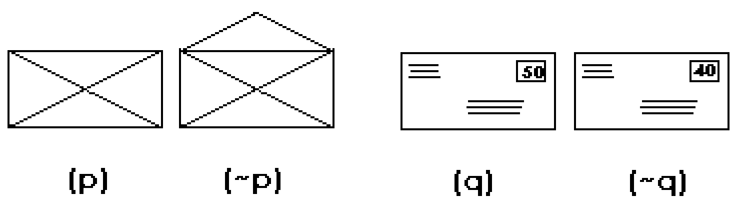
Subjects are expected to choose the case related to an equivalence relation, i.e. the if q then p case.

If the subject had difficulty or failed to identify the second card to the Wason's Card task, then the observer continued with the Johnson's Envelopes task, otherwise task 3 would have been offered.

Task 2: Johnson's Envelopes

This task is almost identical to the first except four especially prepared envelopes are used instead of cards as follows:

Figure 2: Johnson's Envelopes



This is another object based task based on the work of the British psychologists Johnson, Laird, Legrnzi, Oakhill (Girotto, 1989, p. 205), most recently in 1986. The four envelopes are placed face down on the desk top and the same introduction and explanation as Wason's Cards is given along with the following if-then statement to verify or "control for" (Girotto, 1989, p. 197): If the envelope is sealed, then it has a stamp of 50 cents on it.

Correct Solution

A correct solution would be choosing the case of p and ~q, i.e. the sealed envelope and the envelope with the 40 cent stamp.

Purpose

The main purpose is similar to Wason's Cards: "Can the subject identify the correct solution? What justification does the subject give?"

The secondary purpose is a comparison of the Wason's Card task to Johnson's Envelopes task. Are subjects much more successful at and/or are they much more able to deal with the Johnson's Envelopes task ? What differences and similarities exist with the justifications offered for the two tasks?

Expectations

In this task the if-then statement is practically identical to the first. What is different is the lack of "vowelness" or "eveness". Since this task is slightly more concrete, and the literature suggests a higher rate of success with this task, one would expect that a subject can fail at Wason's cards and succeed at this task.

The difference of complexity is in the specific mathematical reasoning required to resolve the Wason cards problem, while everyday reasoning would be sufficient to resolve the envelope problem. This difference is enough to change the expectations from the dismal 10% to a reported 90% (Girotto, 1989, p.197).

We note here that the expectation of this task should be slightly reduced with non-European subjects. There exist a cultural bias as Europeans are used to a "printed matter" option in their postal service that carries a reduced rate. Thus, for a European, the idea of having different value stamps on sealed and non-sealed envelopes is "every-day" and ordinary.

Task 3 and Task 4: Modus Ponens with no mathematical content

The subjects were given the two following sets of statement and were asked to give the conclusion that they thought would fit best or was most appropriate:

(task 3)

If you are exercising every day, you are fit.
John exercises every day.

(task 4)

If you do not study, you will fail.
Peter is not studying.

Correct Solutions

Task 3 has the solution “John is fit.” and task 4 “Peter will fail.”.

Purpose and Expectation

The statements are essentially the propositions of valid syllogisms of the form modus ponens, with the conclusions missing. The purpose was to confirm that the subjects understand a simple material implication set in simple ordinary language. The subjects were expected to give a correct solution without difficulty.

Task 5: Fallacy of Affirming the Consequent

The subjects were given the following set of statements and were asked to give the conclusion that they thought would fit best or was most appropriate:

If it is raining the streets are wet.
The streets are wet.

Correct Solution

Stating that there is "no appropriate conclusion" or simply avoiding committing a fallacy. Not being able to give any response can also be considered as an appropriate solution if the justification suggests that the subject is unwilling or concerned about committing a fallacy.

Purpose

The logical sophistication of a subject might be evaluated by observing their responses to tasks that are meant to trap them into committing logical fallacies. This task was meant to trap the subject into committing The Fallacy Of Affirming The Consequent . The statement is non-mathematical and is leading by the context in ordinary language. It is common sense to assume that it had rained on finding that the streets are wet. It is expected that a number of subjects will commit the fallacy, but not a majority.

Subjects who identify an if q then p case in task one are expected to fall for this trap easily.

Sophisticated subjects can be expected to use every day situations as counter-examples. For example, the passing of a street cleaning machine can also explain wet streets. Extremely sophisticated subjects can be expected to state that no appropriate conclusion is possible or maybe simply to identify the trap set: "You want me to say, 'it was raining', but that would be wrong (fallacious, the Fallacy of Affirming the Consequent)". The wording and justification offered by the subject can easily give one an idea of the level of sophistication.

Note

Once it is evident in any of the tasks, from task 3 to 8, that the subject is unable to cope with the level of difficulty, we would jump to task 9, concerning finding logical flaws.

Task 6: Fallacy of Affirming the Consequent with mathematical content

The subjects were given the following set of statements and were asked to give the conclusion that they thought would fit best or was most appropriate:

> If x is a square root of a non-zero number then x is a positive number
> 3 is a positive number.
> ______________________________

Correct Solution

For this task, the justification and not the solution is important. If a person concludes that 3 is a square root of a positive number ($3=\sqrt{9}$), he or she may not be committing a fallacy. The subject may be stating a mathematical fact. The main problem with this statement in Task 6 is that it is false.

Purpose and Expectations

This task is similar to task 5 in that it is also a trap. Unlike task 5, one would not expect students to commit a fallacy simply because there is the mathematics to sort out within the body of the statements. One would expect confusion and difficulty in culling the mathematics from the logic, neither supporting a strong conclusion.

In this task one might expect subjects to fail to commit the fallacy and demonstrate complete confusion or inability to justify any response. Maybe the subjects will use or study the logical structure more closely and be less likely to commit the fallacy. Possibly, the subject might simply use their mathematical knowledge to determine the appropriateness of a conclusion.

Task 7: Modus Tolens with mathematical content

The subjects were given the following set of statements and were asked to give the conclusion that they thought would fit best or was most appropriate:

> If a is a prime number, then a is divisible only by a and 1.
> b is not equal to a or 1, divides a
> ___________________________________

Correct Solution

The correct solution is that a is not a prime.

Purpose

The purpose here is to further test the level of sophistication of the subjects logical ability using varying levels and types of questions. This logic is quite complicated and the mathematical content is leading and complicated. The justification was of primary interest in this task. Was the student stumped by the logic, the mathematics or simply the mixture?

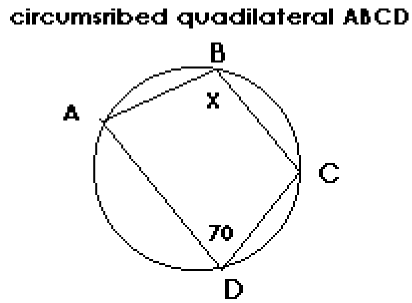
While “prime” and “divide” are not complicated ideas, “b is not equal to a or 1, divides a” is a complicated compound mathematical statement.

Expectation

It was expected that even the teacher subjects would have difficulty with this task.

Task 8: Modus Ponens with mathematical content and geometrical figure

If a quadrilateral can be circumscribed by a circle, then opposite angles are supplementary.



Here is a quadrilateral circumscribed by a circle.

Correct Solution

The correction solution here is that the angles x and 70 are supplementary, i.e. $x + 70 = 180$, therefore, $x = 110$.

Purpose

This task is another modus ponens type syllogism without conclusion. The purpose of this task is to evaluate the added feature of a geometric diagram.

Expectation

All subjects were expected to do well since they have all been recently exposed to these types of objects in their math class. The reasoning was expected to be informative as this is not something covered in class. Note that this task was not utilized.

Task 9: Searching for "logical flaws"-- ordinary language and reality

The subjects were asked to find the flaw in the following argument:

A congressman once was accused by a television news commentator of having accepting a bribe. The accusation was refuted and the commentator apologized:

"I apologize for the misstatement I made concerning the fact that the congressman was guilty of accepting a bribe. "

Correct Solution

Identifying the words "the fact that" as renewed accusation.

Purpose

In this task, the apology is loaded in such a way as to re-accuse the commentator, i.e. "the fact that ". This statement certainly was not intended as a sincere apology. The purpose is to check on the subject's ability to pick up on the loaded language.

Expectation

The language and meaning, plus that the subject is asked to "find the flaw" makes this task direct. All subjects are expected to recognize that the apology is empty, but perhaps some subjects will not be able accurately identify the problem or explain why it is a problem.

Task 10: Accepting/rejecting a universal affirmative: The counter example

The subjects were asked verify if the following statement was **always true:** For any whole number x, $x^2 + x + 41$ is a prime number.

The subjects were given every opportunity to reject the universal although the observer purposefully suggested they first check some "examples" . The observer went as far as to ask: "When can this statement be false?".

Solution(s)

a) No number of examples would be adequate, since only one counter-example would supply proof that it is false.

b) At least one counter example exists, when x =41, therefore the statement is not true.

Purpose

The purpose was to see if the subject would accept examples, even many examples as “proof” or “evidence” that the statement was true. A sophisticated subject should recognize that no number of examples would be adequate and/or one single counter example would prove the statement false.

Expectation

Senk (1985, p. 447) and Fischbein (1982, p. 131) and my experience with an earlier project suggests that most subjects would fall into the trap of affirming that the statement is true.

Task 11 and Task 12: Two Proofs

The subject was asked to prove one and/or both of the following statements about sums and product of even and odd number:

(Task 11) Prove that the sum of two odd numbers is an even number.

(Task 12) Prove that the product of two odd numbers is an odd number.

Correct Solutions

Any diagram or justification is acceptable here. The correct solution is not as important as the question of whether the subject will even be able to attempt a solution and what this attempt resembles.

Purpose

Few students or teachers are happy or comfortable when asked to produce a proof. The design of this task was meant to draw out the ability of the subject in demonstrating or justifying a simple mathematical statement. The statements were selected because they are quite simple to understand and of a basic mathematical content: odd/even numbers and sum/product. A number of justifications are possible, including diagramming, allowing for both formal and very informal approaches.

Expectations

No subject was expected to immediately give a formal proof. Instead, subjects were expected to fall on some type of diagrammatic or verbal argument, reviewing cases or possibilities. The following dot diagram has been used by educators to demonstrate relationships of even/odd numbers and perhaps would have been likely expectation:

Possible Proof of Task 11 Offered by a Subject

Numbers represented in eveness or oddness by pairing the even parts as dots.
An even number is represented by the ending of two dots, •• , i.e. 6 is •• •• ••

An odd number is represented by the ending of only one dot, •, i.e. 11 is •• •• •• •• •• •

Thus any odd number ends with one dot, • and any even number with two dots, ••

The sum of any two odd numbers ends is

Any number of pairs of dots, •• and only on e • for first number

Any number of pairs of dots, ••, and only one • for second number

So, the sum would be some number of pairs, •• and two lone dots, • • somewhere.

Put together, this would always give paired dots and no single dot •• •• •• ... •• •• •• ••, meaning the sum is always even.

Sophisticated subjects were expected to build up a more formal justification after studying one or two cases. The initial part of the proof might start with, let **a,b** be any distinct odd numbers, x, y $\in$ Z, and then stating something like :

Note that **a** can be written as 2x +1, and **b** an be written as 2y + 1,

so **a**+ **b** = 2x+1 + 2y+ 1 = 2(x + y) +2 = 2(x + y + 1), an even number.

An inability to provide any adequate or organized justification for student subjects would not be surprising***.***

Task 13 and 14: Mathematical Definitions in non-standard form and language

The two following definitions were offered to the subject.

(Task 13) Definition - Fitoalt

1) x,y are real numbers
2) A mapping is such that a pair of coordinates are " mapped" from (x,y) to (x`, y`): (x,y) $\rightarrow$ (x`, y`)
3) A mapping is a "Fitoalt" if the mapping is unique for (x`, y`), that is to say, the mapping has only one unique (x`, y`) for any given (x,y).

Consider the mapping (x,y) $\rightarrow$ (x + y, x-y). Is this mapping a "Fitoalt"?

(Task 14) Definition -Tigger

A "Tigger" is a collection of pairs of numbers with the following property:
If (a,b) and (a,c) are both in the collection, then b = c.

Consider the following collection of pairs of numbers: 1,1 6,36 -1,1 -1/2,1/4 -2,4 3,9 10,100

Does this collection satisfy the definition of a "Tigger"?

Solutions

Task 13 is a Fitoalt, i.e. this definition fits the concept given: a function in terms of a linear transformation f-i-t-o-a-l-t. The mapping given is a simple linear transformation and the definition is satisfied.

In Task 14 the definition is satisfied. This given set is are some of the order pairs from the function $f(x)=x^2$. The definition is a relatively common definition found in many algebra textbooks, but usually related to a visual "straight vertical line test" for a function rather than an analytic test.

Purpose

Both of these tasks are definitions mired in unusual mathematical language with uncommon or made up terms: "Tigger" (the name of the tiger character from the Adventures of Winnie the Pooh) and "Fitoalt" (acronym for "function in terms of a linear transformation").

These tasks represent new definitions for almost any subject and are used to explore the options a subject will chose to understand a new definition.

Expectations

Subjects are expected to either not attempt the task or, hopefully, they will fall onto some other method. It is expected that the subjects are likely to attempt to understand the definitions using the logical structure of the statements. The subjects will probably attempt to “unpack” the logical structure of the statement.

Responses to the tasks

In this section we make a summary of the subjects responses to the tasks. In general, it was fairly simple to ascertain the subject's sophistication with logic from the first few tasks. Subjects who did poorly on the first task seemed to do poorly throughout.

Responses to Task 1: Wason’s Cards

Controlling the rule proved difficult as each subject had some difficulty with this task. All subjects identified the case p implies q. Only one subject, Maria, identified the counter example after some directed questioning, as expected.

The justifications for the choices were predictably confused and not easily vocalized (Girotto, 1989, p. 196, Fischbein, 1982, p. 131). In one case the subject chose only the one if p then q case. In the two remaining cases chose the if q then p, i.e. a logical fallacy. This suggests that these subjects would accept an argument based on the fallacy of affirming the consequent.

Responses to Task 2: Johnson's Envelopes

Subjects seem more at ease with Johnson's Envelope task and did better at this task than the Wason's Card task. The justification was "fuller" with this task. In addition, subjects were quick to become unsure of the appropriateness of their responses as they gave justification. For example, as Anne is trying to justify her selection of the 50 cent envelope she essentially justifies the 40 cent envelope as the appropriate solution. She recognized the problem of selecting the 50 cent has she tries to justify it as a solution. She does not even blink justifying the similar wrong solution of choosing the even number in task

Responses to Task 3 and Task 4: Modus Ponens In Ordinary Language

All subjects gave the correct solution without difficulty as expected for both tasks.

Responses to Task 5: Fallacy of Affirming the Consequent

Some subjects fell for the trap. Even so, they seemed uneasy or uncomfortable about it. Shawn, for example, realized "there may be a problem" but 'it is raining' "just makes sense". The same subjects who chose the if q then p for task one also fell for the trap in this task, as predicted.

Responses to Task 6: X Is A Square Root

This task was meaningless to the student subjects and confusing to both teacher subjects. These responses suggest that this task was poorly written, or not well placed and/or too complicated.

Responses to Task 7: Prime Number

Maria was the only subject who did this task but even she was unclear about where to go with the statement.

Responses to Task 9: Congressman Flaw

Three of the four subjects recognized that the statement is not intended as an apology and were able to articulate the problem as the key words “the fact." The remaining subject stated that the commentator was merely apologizing and did not see any flaw in the argument.

Responses to Task 10: Prime Number

The sophistication of the subjects was well demonstrated by this task. The subject (Maria) showing both strong mathematical and logical (ordinary language) background immediately pointed out that example are good for verifying case by case only. She immediately suggested that a proof was needed to confirm the rule universally and that an example would be best used for a counter-example.

Shawn, who demonstrated a moderate ability with the tasks, was also unwilling to accept the statement even after some examples. Shawn’s attitude and weariness demonstrate his worldliness when being asked questions in school. He knew he was being “set-up” from the previous task. “It would only take one number to prove it wrong," was his response when being pressured to commit to accepting the statement based on examples only.

Responses to Task 11 and Task 12: Proofs

Every subject believed the statement to be true. Each subject's response seemed to match the same sophistication demonstrated in the previous tasks.

The most striking response or statement was Maria's when confronted with the task 11: Proving the sum of any two odd numbers is even. Maria indicated that the task was different from the others with her statement "Got to get my math mind in here."

Even after two examples, Maria was dissatisfied and simply started with "let x be any odd number" and went on to construct an appropriate proof of the statement.

Mike's response matched his success in most other tasks, that is very weak. After five examples Mike considered the statement proved.

Shawn's was convinced but not because of examples. He worked out two examples and then stated "I can do this with any two odd numbers," meaning he could generalize the method.

Responses to Task 13 and 14: Definitions

Only the subjects Maria and Anne (the teachers) were able to respond appropriately to the definitions. Maria was not daunted by the unusual language and words and went on to identify both definitions simply by resemblance.

Maria does fall onto the logical structure to understand the definition "Tigger." In addition she states a preference to ignore the "example" and understand the definition on its terms:

Marc: "What do you find confusing here?"

> Maria: "I am looking at these numbers and I am trying to think back at what I learnt ... trying to use if and only if [logic].. I would rather have the definition and not the numbers at all

Conclusions From The Observations

The first comment needs to be about the behavior exhibited by the observer. With so many issues to deal with, plus the inherent complexity of keeping track of each task, the observer was easily sidetracked within tasks and failed to acquire an appropriate image of each subject's thinking for some tasks. Reading the transcripts and watching the video segments suggests that many missed opportunities in pursuing certain tracks of the subject's thinking.

This shows a weakness in the design of the observations: a prepared check list of observations and questions should have been used by the observer to facilitate adequately completing a task. In addition, using a set of two or three shorter observations instead of one fairly long observation would allow some follow up.

Note that a set of fixed tasks and questions were at first considered but then rejected. For this observation, the freedom of choosing among tasks, allowing an open set of questions and allowing different levels of participation in the task was felt to be a necessary approach for this initial observation. The purpose was to get some idea of the thinking and not necessarily to test any preconceived conjectures.

In chapter II we posed a number of question about the relations between logical thinking and proving. Lets us consider these questions on the basis of the observation.

In general, the subjects' behavior suggested that the addition of mathematics objects complicated understanding. This was the case even for the observer. Subjects did better at Johnson's envelopes than Wason's Cards, so Girotto's conjecture is essentially supported by the observations.

At least some of the tasks are mathematical activities. From the observations, it seems obvious that some logical thinking is needed for success. Subjects tended to carry over success, or lack of success, from ordinary language tasks to the more mathematical tasks. Thus, the conjectures of Seldin and Seldin are also supported.

That is to say, those who did well at general logical thinking also did well at the more mathematical activities. Those who did poorly at general logical thinking also did poorly at the mathematical activities. There seems to be a connection between success at non proof type mathematical activities and logical thinking.

Unfortunately, it is difficult to determine if this is simply a difficulty of logical thinking about ordinary language being carried over into mathematical activities that are necessarily written in a more formalized language.

This project's observations also seem to confirm a strong relation between difficulty with language and difficulty to deal with both logical statements in ordinary language and logical statements written in language with mathematical components. It is not surprising that reading and comprehension skills seem to have an impact on one's ability to "unpack" the logical contained in a linguistic statement.

Does the ability to think logically entail the ability to do mathematics, or is there more to mathematical thinking than just logical thinking? What would that be? The behavior of both Anne and Shawn suggest this might be so.

Shawn is a quick student and was fairly capable at ordinary language logical abilities. Adding a little mathematics easily short circuited his thinking. Anne, on the other hand, is at ease doing and teaching high school mathematics, yet she fell for the same types of logical traps as Shawn.

This state of affairs suggests that there must be something more than logic to be successful at mathematics, at least at this level. It would be interesting indeed to compare the ordinary language logical thinking of university mathematics students and compare this to their mathematical logic abilities.

Does the learning of mathematics improve one's logical thinking? While there is no evidence of this directly, Maria's performances suggest that a good basis in mathematics is advantageous when tackling logical problems.

Chapter IV

Conclusions

As an educator, especially one who is interested in how students might think 'logically', I have developed a healthy awareness of either oversimplifying the notion of thinking of mathematics as logic, or of over emphasizing the relationship between logic and mathematics.

If anything, I have come to think along the lines similar to Putnams' thinking: that the science of mathematics is strong enough to not need the foundations searched for. In as much as a foundation for mathematics might be thought to be necessary to give mathematics stability and/or basis, the science of mathematics, I think, is quite strong as it is.

Tymoczko's notion of mathematics as an art adds a whole new dimension to the philosophy of mathematics. Certainly it suggests that there must be something more than logic to be successful at mathematics. This "humanistic" quality of mathematics deserves exploration and perhaps will enlighten us on better approaches and improved learning to the teaching of mathematics

The "whole language" debate is going to have an impact on the teaching "proving". Is it possible that teaching proof is like teaching grammar? Will it take a "whole math" approach to help students acquire an adequate level of ability in logical thinking? The linguistic component of logic is an important factor that should be explored.

Some of our subjects were relatively unsophisticated in both language and mathematical skills and we did not attempt such an exploration.

It also might be interesting to explore Piaget's notion that maybe some students still have not developed a cognitive base on which we can adequately add higher logical reasoning.

The observations did suggest that some tasks can become effective tools for helping students acquire logical skills and that some cognitive obstacles might exist. Perhaps we can develop more successful methods to improve students' logical thinking in mathematics. Which questions will answer our difficulties? Will it be a linguistic comparison, a "whole math" school of teaching, a humanistic trend in mathematics education, or will it be simply a set of more appropriate tasks of many sorts that support logical thinking in mathematics?

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