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Continuity and The Infinitesimal as Cognitive Obstacles in Modern Philosophy

Course Paper for

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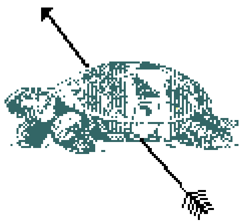

"You haven't seen a tortoise up there, have you? Damned fast things, go like greased thunderbolts, there's no stopping the little buggers."
"Tortoises?" Tepic said. "Are we talking about those, you know, stones on legs?".
"Fastest animal on the face of the Disc, your common tortoise," said Xeno, but he had the grace to look shifty. "Logically, that is", he added.[1]

This paper will explore the hypothesis that the concepts of "continuity" and "the infinitesimal" form a cognitive obstacle in modern philosophy. An explanation of a cognitive obstacle will be established followed by an exposition of the historical basis for a continuity-infinitesimal cognitive obstacle, from the ancients to the pre-moderns. An analysis of some of the representatives of the three main schools of philosophy of the modern period, the rationalists, the empiricists and Kant will be given in terms of this hypothesis.

Cognitive Obstacles

> The single most important factor influencing learning is the knowledge already possessed by the student. Ascertain this and teach accordingly.[2] David P. Ausubel

[1] Terry Pratchett, Pyramids, New York: Roc-Penguin, 1989), p. 186-7. Tepic finds Xeno and his partner, Ibid are testing out Xeno's claim that the arrow will never reach the tortoise at the Axiom Testing Station. The "logically unexpected" result is depicted.
[2] in Bargellini, Overcoming Some Childrens' Cognitive Obstacles In Chemistry : A Research In Primary School On Concepts Of Interaction, Conservation And Transformation. Università Degli Studi Di Pisa, Dipartimento Di Chimica E Chimica Industriale Dell'università,
Via Risorgimento, 35 - 56126 - Pisa (Italy) Research And Development Centre 2 (p. 1)

Jean Piaget claimed that knowledge is the basis on which new knowledge is added by the processes he called accommodation and assimilation. Accommodation occurs when a concept or schema is modified by experience of new events (change). Assimilation occurs when a new event is included into an existing concept or schema (addition). When a new piece of information conflicts with an existing one, a epistemological conflict is introduced that has to be resolved through these processes resulting in a new set of schema that form one's conception of the world. More complex events result in a more complex series of changes and adjustments to a person's schema. In some cases, the new event does not fit well in the schema. The epistemological difference is too great to be assimilated or accommodated. It is the existence of present knowledge that creates this situation and so is not simply a matter of repeating the lesson or trying to adjust to the new event. This problem is called a cognitive (or epistemological) obstacle. Nicholas Herscovics stated the problem as follows:

> A piece of knowledge that has in general been satisfactory for a time for solving certain problems, and so becomes **anchored** in the student's mind, but subsequently that knowledge proves to be inadequate and difficult to adapt when the student is faced with new problems" [3]

The existence of a cognitive obstacle can be quite evident. In a teacher-student situation, the teacher finds themselves repeating and repeating explanations with alternative approaches but the student remains frustrated and simply "does not get it." The "tool" and "concepts" seem to be available but the person is simply unable to make the cognitive step of understanding.

An example in teaching algebra is the step from writing equations of lines using y = mx + b to writing lines in function notation using f(x) = mx + b. If the time students use the

[3] Nicholas Herscovics, "Cognitive Obstacles Encountered In The Learning Of Algebra", in S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra*, (Erlbaum Publishing, New Jersey, 1989) pp. 60-86

y = mx + b is very long and never includes examples in the function notation, it will be difficult to get students to adjust to the new concept. The structure of mathematical knowledge is built from the particular to the general, the simple to the complex. Thus, it is very prone to these cognitive obstacles. Poorly trained teachers are most often unaware of the issues involved in the schemes they build in their math students. This results in poor learning and frustrated student who come away with math phobias and a general hatred of mathematics and science (misogamy) . The best way to deal with a cognitive obstacle is to carefully build schemes without creating the obstacle. i.e. to structure learning situations that avoids or reduces them, rather than dealing after they have been constructed.

The early conception, strengthened by continual examples and evaluations is very strong in any students conceptual thinking. To refute and change this basis is extremely difficult, as is evidenced by acute failure in teaching college students mathematics. Given this failure, it is no surprise that one finds that continuity and the infinitesimal are cognitive obstacles among today's students learning the calculus.[4] Imagine how this might impact thinkers of the modern age. For many years they struggled with concepts. Suddenly, they are exposed to new ideas of continuity and the infinitesimal, idea not fully developed and not fully understood by anyone of the time.

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[4] For example, see **John Monaghan**, "Problems with the language of limits", *For the Learning of Mathematics,* (1991), *11*-3, pp. 20-4; Lisa D. Murphy, *Students Conceptions of Rate of Change*. (1999). (In progress.), **Anna Sierpivska**, "Humanities students and epistemological obstacles related to lim its", *Educational Studies in Mathematics,* (1987). vol. *18*, pp. 371-97 and/or **Steven R Williams**,. "Models of limit held by college calculus students", *Journal for Research in Mathematics Education,* (1991). *22*, pp. 219-36.

Historical Evidence Of A Problem With Continuity And The Infinitesimal

Everyone knows it was that trouble maker Zeno's fault. Any consideration of problems with continuity and the infinitesimal must start with the paradoxes of motion and time given by Zeno of Elea (488?-450 BC). These paradoxes we given in defense of Zeno's teacher, Parmenides, and were meant to support he notion of the world as unchanging unity and persistence of being. Not surprising that one of the first metaphysical problems in recorded history is one of continuity and the infinitesimal. The importance of these paradoxes is illustration by how well they confound thinkers, even people of our own time. Let us consider a version of the paradox and solution.

Consider a turtle and a person armed with a bow and arrow. If one lets the turtle go a distance, call it distance d, and then one aims the arrow at the turtle and lets off, the normal expectation is a turtle kabob. But consider that the arrow must travel from launch point to the turtle through some particular trajectory of distance d. (Geometrical set up)

Before the arrow can go all the way, it must travel at least half way. Before the arrow must travel half way, it must travel one half of that sub distance, or a quarter of the distance. The next sub distance then should be one eighth and so on. This means that one could partition the distance into many sub distances with measures as follows:

$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots+\frac{1}{n}$, an analytical statement of the geometry. This series can be extended letting n go to infinity. In other words, there exists an infinite number of finite, but measurable, distances that partition the distance between the idiot with the arrow and the turtle. The sum of these partitions, an infinite number of finite but measurable segments (geometry of number), no matter how small, must be infinite (sum of geometrical number). The distance from the arrow to the turtle is therefore infinite, the arrow cannot then reach the turtle in a finite period of time. Any volunteers to be the turtle?

The solution is relatively simple since the series given is Geometric (not as in geometry but the name of a particular type of sequence). Each term is ½ times the previous term and so the nth term can be expressed as $a_n = a_1$ r, where ½ is the first term or a_1 = ½, and the common ratio is 1/2, $r = \frac{a_{n+1}}{a_n} = \frac{1/4}{1/2} = \frac{1}{2}$. The sum of a geometric sequence[5] can be proven inductively to be given by $S_\infty = \frac{a_1}{1-r}$, in this case $S_\infty = \frac{1/2}{1-1/2} = 1$. In other words, this infinite series of sums converges to 1, the unit distance from the arrow to the turtle. Turtle kabob.

Notice how the paradoxes are directed at showing that the "unlimited or the continuous, cannot be composed of units however small and however many ".[6] This is weighted against the Pythagorean idea that set forth a world more abstract, "postulating that number in all its plurality was the basic stuff behind phenomena"[7], an idea of multiplicity and change. In addition, note that the associated questions "What is meant by the **infinitesimal?"** and "What is this concept **continuity** that makes it different from contiguous?" are mathematical as mush as they are philosophical. Irrespective of a rational, empirical or Kantian foundation, any philosophy with a

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[5] Probably not well explained until the time of Jacques Bernoulli (1654-1705) who regularly shared ideas with Leibniz.. Even so, note that it is clear that Bernoulli went far beyond Leibniz's understanding of convergence of series. See Boyer pp. 465-9.
[6] Internet Encyclopedia of Philosophy, James Fieser, editor, http://www.utm.edu/research/iep/z/zenoelea.htm
[7] Carl B. Boyer and Uta C. Merzbach, A History of Mathematics, second edition, (New York, Wiley, 1989) p. 85

metaphysical/epistemological consideration will have some spin on the questions of infinitesimal and continuity.

Indeed, without some non-geometrical analytical sense of the infinitesimal, one cannot deal adequately with the paradoxes of motion. This is obviously going to a problem for the pre-modern philosophers who lack adequate understanding of the infinite but who set the stage. The algebra of the Middle East, which blossomed with Mohammed Ibn-Musa al-Khwarizmi's (800-850 AD) book *Al-Jarbar*[8]*,* seems to be among the first texts in the history of western philosophy that introduced a non-geometrical analysis that continental mathematicians and philosophers adopted as much as any *renaissance* of Greek thinking.   As a result, one can identify some of the first non-geometrical speculations in the infinitesimal turn up late in the middle ages.[9]

The problem is that the geometrical viewpoint was held for a very long time, and not just by one person, but by society and culture. In the same way that a notion held for a long time in one person can lead to a cognitive obstacle, the culture of mathematicians and philosophers hold a schema of knowledge that potentially posed a cognitive barrier to either accepting or making use of the new concepts of continuity and the infinitesimal. The geometrical notions and problems related to Zeno's paradoxes were too useful and shaped thinking. Let us consider this in light of some of the scholars of the pre-modern age.

Thomas Bradwardine (1290-1349) wrote *Geometrica speculativa* and the *Tractatus de continuo* in which he "argued that continuous magnitudes, although including an infinite

[8] A foundation work on algebraic equations, but does still have a good deal of deduction from a geometrical basis. For example, square root is geometrical: It is asked "a square with area 25 have want magnitude of sides, answer 5. Hence "square root" or "square base."
[9] Archimedes, along with some late Greeks and other Romans also developed some thinking in the no-geometrical, but it took a grounding in algebra to bring this the abstract level.

number of indivisibles, are not made up of such mathematical atoms, but are composed instead of an infinite number of continua of the same kind"[10]. Bradwardine was attempting to introduce some new concepts but is dogged by speculations on the continuum. This issue was popular among scholastic scholars such as St. Thomas Aquinas and later influenced the Cantorian infinite of the nineteenth century. But Bradwarnine's argument is similar to one of the Eleatics. For Bradwarine, numbers are represented geometrically as segments of lines such that "the unlimited or the continuous cannot composed of unit however small and however many".[11] Zeno's concept of the infinite is causing a problem in the transition to a analytic mathematics and physics.

Simon Stevin (1548-1620) is another example. In his book *Statics* of 1586, Stevin used the Archimedean principle to show a demonstration by numbers in which a sequence of numbers tended to a limiting value."[12] This is directly related to the Pythagorean idea of using "the property of continuous magnitudes" as a thing apart from number." [13] To this Galileo Galilei (1564-1642) asserted "that infinites and indivisibles "transcend our finite understanding, the former on account of their magnitude, the latter because of their smallness; Imagine what they are when combined." This is not far from Immanuel Kant's (1724-1804) solution. That is to claim the paradox beyond or transcendent to possible knowledge.

The tone changed with Bonaventura Cavaleri (1598-1657) who started to consider not the geometrical but numerical notion of the infinitesimal that lead in part to Newton's and Leibniz's conception of calculus. More importantly , in terms of mathematics,

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[10] Ibid., Boyer, p. 294
[11] Ibid., IEP-zenoelea.htm, p. 1
[12] Ibid., Boyer p. 360
[13] Ibid., Boyer p. 87

Cavaleri's introduced the analytic geometry of the infinitesimal that lead to a rigid analysis by Augustin-Louis Cauchy (1789-1857), appearing a full century after Newton.

> Nature and Nature's laws
> lay hid in night;
> God said, Let Newton be!
> and all was light. (Alexander Pope)
>
> But Mathematics and Metaphysical laws
> lay hid in infinity,
> God said, Let man understand!
> and all was Cauchy. (M. Corbeil)

In *Résumé Des Lecons Sur Le Calcul Infinitesimal* (1823) *And Lecons Sur Le Calcul Différentiel* (1829) Cauchy "made the limit concept of D'Alembert[14] fundamental, but gave it an arithmetic character of greater precision."

> When the successive values attributed to a variable approach indefinitely a fixed value so as the end by differing from it by as little as one wishes, this last is called a limit of the others.[15]

As Boyer points out, "Where many earlier mathematicians thought of an infinitesimal as a very small fixed number, Cauchy defined it clearly as a dependent variable." This is the shift from a mostly geometrical foundation for the calculus to a analytical one. With Newton and Liebniz, the area under a curve f(x) is given by a definite integral by equal successive rectangles with base "b – a" under the curve. By taking very small but equal intervals of x for this sub-base, one can approximate the area under the curve by successive calculations of rectangles, i.e. with sub-bases Δx. The idea that one can do this infinitely as Δx gets very small is the basic idea of the calculus:

Newton - Leibniz Area = $\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x$ where Δx = (b-a)/n The subintervals are all of equal length. But the Riemann[16] sum with subintervals of **unequal length** which seems to

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[14] Jean Le Rond D'Alembert (1717-1783)
[15] Ibid., Boyer, p. 575
[16] Georg Friedrich Bernard Riemann (1826-1866)

appear just after Cauchy's definition is given by $\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i$ where $x_i = i^2/n^2$ and Δx_i is the width of the ith interval. The upshot seems trivial but shapes the problem. The mathematics of Newton's physics does not force the issue of the infinitesimal, the infinite, nor the limit. No modern philosopher, then, had seen how the tools to adequately deal with Zeno'a paradoxes applied to the problem. Their training and understanding was invariably from the physics or Newton or the Elements of Euclid. The late and incomplete non-geometrical approach challenged their conceptions which them selves were long held, not only on a personal basis but also on a wide cultural basis, and thus may have formed a cognitive obstacle to seeing the extension into their own metaphysics.

Modern Problems

Francis Bacon (1561-1626) recognized a first step in science by using induction in negative sense. He realized that you could not verify all possibilities of a generalized claim, even if the claim considered only a large number of finite elements. "In establishing any true axiom, the negative instance is the more powerful".[17] Thus Bacon sets forth not only the idea of verification but introduces the idea of falsification. That science should aim to attempt to falsify a universal claim, a distinction between enumerative (finite positive) induction versus eliminative (counter example negative) induction[18]. But what Bacon missed, and Thompson did not identify, is consideration of induction in very large sets or the infinite. While Bacon clearly saw the strength of induction, he confused the principle of well ordered sets, especially with the very large or very small, and thus could not see how

[17] Francis Bacon, quoted in Garett Thompson, Bacon to Kant, (Waveland Press: Illinois, 2002) p. 122
[18] Ibid., Thompson, p. 122

one might perform induction on either very large sets or infinite sets. Bacon lacked a conception of the infinitesimal but had very strong idea of induction that hindered thinking in terms of the infinitesimal.

For Rene Descartes(1596-1650), the undisputed father of modern philosophy, "The entire universe … [is postulated to be made up] of matter in ceaseless motion in vortices, and all phenomena were to be explained mechanically in terms of forces exerted by contiguous matter"[19]. Descartes understood that the universe is atomistic or discrete in parts and everything is explainable in terms of matter (or extension) in motion. While his philosophy and the *Discourse de la methode* is an original attempt at rationalist self-reflection and doubting, the mathematics of Descartes is linked to earlier traditions.[20] Thus, the rationalistic starting point of the self and the problem of getting out of a Cartesians subjectivity will also have an impact on problems with the infinite. If the essence of man is reason, and the epistemological center is the individual, and since this is utterly unextended in Cartesian philosophy, then it will be very hard to even conceive of the infinite understandable and knowable by man. Descartes will be able to extend hard science and mathematics in the engineering sense, but this will only help to build onto the cognitive obstacle. The influence of Decartes then is negative and platonic: pure mathematics where the concept of numbers are cleared away to give room for the purely abstract and Parmenidean idea of number, along with the misunderstanding of the infinitesimal.[21]

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[19] Ibid., Boyer, p.375
[20] Ibid., Boyer, p.375
[21] A nature of world must be some kind of pure mathematical and geometrical structure where matter is pure geometry (not number) as evidence by reference to Archimedes and Newton's first law in Med II 1st passage.

Gottfried Wilhelm Leibniz's (1646-1716) philosophy can be viewed as a search for a rational-logical foundation of philosophy and here that one finds one of the best examples of the cognitive obstacle. Leibniz saw or envisioned the idea of deducing math (and physics) from logic. His scholastic training, especially in Aristotle, shows up in his belief that the categorical (propositions and syllogisms) are absolutely fundamental as a basic theory of propositions. To Leibniz, the subject-predicate form of propositions is basic and all meaningful propositions can be reduced to this form. The fascinating aspect is how Leibniz thought in terms of relationships in an almost functional-mathematics way. It really seems that he understood much of metaphysics in terms of mathematical ideas and Newton's physics.

An example can be found in consideration of "Necessary and Contigent Truths" in Opuscles et fragments indits de Leibniz[22]. "An absolutely necessary proposition is one which can be resolved into identical propositions …" is an argument given and then illustrated by an example of The Prime Factorization Theorem in terms of a discrete consideration. All propositions of the subject-predicate type are defendable by using a relationship that is functional. Leibniz seems to force a discrete rather than a continuous viewpoint as is want of purely atomistic metaphysics. The monad is "nothing else than a simple substance .. having no part, cannot be decomposed".[23]

Leibniz understood four basic types: bare monads; unconscious and form unextended, animals; some degree of memory and discrimination in perception, rational soul; self-conscious and apperception/reasoning and the God Monad; the actualization of

[22] T.V. Smith and Marjorie Grene, Philosophers Speak for Themselves: From Descartes to Locke, (University Chicago Press: Chicago, 1940) p. 307
[23] Ibid., Smith, p. 307

the infinite monad or monad of full duration. No universal causation exist between monads although their may be some appearance of this because of “prehension” Continuity then, will be a major problem, especially for the God monad: Like Descartes, Leibniz believed that to deny the possibility of causal relationships between substances seem incompatible with the assertion that monads require the causal influence of God in order to continue [24]

Here’s the rub. If Leibniz is considering relationships as “functional”, then the God monad is the infinite sum, literally the “convergent sum” of all other monads. This is not far from how Leibniz thought in terms of his calculus, especially as compared to the “fluxations” explanation that Newton gave. Note only does this avoid a pantheism by saying God is not a “property” of each monad but is found in each (“prehension”) in the sense they participate in a functional convergence into Leibniz’s God concept, but it is a truly masterful concept and is a staggering point of view for the time.

The monads are the points needed for an infinite process to take place. as if Leibniz is doing calculus on monads. Unfortunately a problem (Zeno’s problem) occurs in relation to the idea of “between any two monads there is another monad”. Leibniz was well aware of this in terms of rational numbers and understood infinite well in this sense. As noted above, he communicated Jacques Bernoulli on the subject. But Leibniz carried the same problem of irrational number into this conception of the infinite, or the incommensurable infinite. As Cantor pointed out, infinite is a process that is unending without reachable bound. In comparison, Leibniz seems to be saying the set of monads that encompass the God monad is “countable” and the count = God. But, if between every pair of monads there is another monad, this set is infinite but the set is not necessarily countable. This is similar to the notion that the set of real numbers in not countable, but the set of rational

[24] Ibid., Monology 51, in Thompson, p. 97 But he denies the occasionalist Cartesian viewpoint

number is countable.   The problem in physics/metaphysics is immediate: How can such a universe be described if the simplest types of set drive one bonkers? Without boundary might mean never ending but in which fashion – space, time, both or neither? What, then, does it mean to say “simple” or “unity of substance?”

Even Leibniz “shucks and jives” with additional details to monads or properties of monads. This physics also seems to deny complete inductions (Hume will deny causation totally) as Leibniz is analyzing the successive point on a circle and denies that they inductively follow one another – one does not cause the next, nor is there “the vulgar hypothesis of influence …” and also denies the occasional Cartesian viewpoint.”[25]   In terms of physics, he is making a direct, and I would believe , purposeful attack on Newton’s physics. (Just like Einstein’s attack). Gravity for example, is explained by Newton as a tendency – as the “attraction of bodies” – i.e. as a causal relationship between monads.

This circles back to the logico-fundational purpose or teleology of the modern philosophers. Since truth of reason, a finite analysis of statements using principles of identity, non-contradiction and excluded middle, all nice scholastic Aristotelians, and truth of fact, contingent , infinite analysis that ultimately require a ‘reference to the free will of God” [26] Truths are necessary and knowable. Finite types are deducible and knowable by processes that humans are capable of doing based on categorical logic. Infinite types are in principle, knowable and we can do a fair job of deducing a limited aspect of these – but only God knows fully. Thus, a sort of Laplacian illusions exist for Leibniz and some other rationalists. At least if some monad (God) knows, then all is knowable. For Leibniz the

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[25] Ibid., Smith, p. 303
[26] Ibid., Thompson, p. 90

illusion is actual – God has all the information and know now! But what are we ultimately capable of knowing? Leibniz leaves us wanting her and I am not sure what he would say. It is not clear that any other of the rationalist could break out of the dualism introduced by Descartes, the dualism between mind and the objects of the world that make Zeno's paradox so obvious a paradox. Perhaps that while Gods know the full story, as an affirmative universal categorical, we might get a subalterns story, some affirmative categorical . This still means our physics will necessarily have incompleteness and holes, our metaphysics will be stilted and problematic. This is the Einstein nightmare – no unification here!

David Hume (1711-1776) and John Hobbes (1588-1679) take off with this "datum' blind to Leibniz's and Spinoza's objection to such language. "Attraction ignores some underlying understanding at the core of Newton's physics. Such a thing could not be, a "primative" concept " must be a compound of other primitives of our human misunderstanding. Frail humans just don't know enough, in essence denying that humans have the ability to rationally deal with Zeno's paradoxes, except for looking to experience and seeing that the arrow does nail the turtle.

> Hume denied the infinite divisibility of space and time, and declared that they are composed of indivisible units having magnitudes. but the difficulty that is impossible to conceive of units having magnitude which are yet indivisible is not satisfactorily explained by Hume.
> [27]

In essence, the empiricists deny the Laplacian assumption that if we had all the data we could KNOW the world. Actualization of the infinite can only mean, at best, the convergence of the sum and not the reaching of an end. This is a error in making a full geometrical analogy and would be likely given that many of the empiricists seemed to less connected to the new mathematics developing from Caleveri to the Bernoullis.

[27]Ibid., IEP-zenoelea.htm, p. 3

Kant

We need to start a consideration of Kant by identifying Kant as a reaction to Hume. Kant agreed with Hume that all knowledge begins from experience but disagreed with Hume's notion of experience could not provide connections with events. "Concepts without percepts are empty, precepts without concepts are blind." Thus, for Kant, Zeno's paradoxes themselves are simple to answer.

> According to Kant, these contradictions are immanent in our conceptions of space and time, so space and time are not real. Space and time do not belong to things as they are in themselves, but rather to our way of looking at things. they are forms of our perceptions. it is our minds which impose space and time upon object. [28]

But such a simple dismal of the paradoxes does not end the question of continuity or the infinitesimal in Kant. The reworking of metaphysics in terms of epistemology results in a whole slew of considerations and problems with the infinite.

> It is only the unconditioned that reason seeks in this synthesis of conditions, which proceeds serially, and indeed regressively, hence as it were the completeness in the series of premises that together presuppose no further purpose.[29]

Kant seems to be saying that boundary is needed even for an infinite regressive series, ie. nothing can start ex nilo. But infinite series can be postulated regressively that easily have no boundaries. The real number set is a prime example.

But more to the point, consider Kant's First Conflict of the Transcendental Ideas[30] . "The world has a beginning in time, and in space it is also enclosed in boundaries." Almost certainly from a conversation with D'Alembert. It is possible that D'Alembert had a better understanding of the infinite than Kant, but unlikely that it was of a very large difference.

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[28] Ibid., IEP – zenoelea.htm, p. 2
[29] Immanuel Kant, Critique of Pure Reason, Translated and Edited by Paul Guyer and Allen W. Wood, (Cambridge Univ. Press: Cambridge, 1998) p. 464 [B444]
[30] Ibid., Kant, Critique, p. 470 [B456]

But Kant describes here an infinitely in terms of points in space, in terms of geometry: ".. up to every given point in time has elapsed, and hence an infinite series of states of things tin the world, each following another, has passed .." . "States" and "following" are key terms suggesting a geometrical structure of passing physical passing from point to point. The only antithesis postulated is no beginning/end, no bounds and infinite in time and space. But two or more postulates should be considered. For example no beginning/end, no bounds but finite in space and time: a non-Euclidean space.

Interestingly enough, D'Alembert was having similar difficulties with geometrical notions of his conception of logarithms as opposed to analytical inconsistencies. Jean Bernoulli was having the same difficulty, but both were soon corrected by Euler[31]. Recall, Cauchy is still more than 50 years in the future, thus suggesting that even thought Kant was adept enough and comfortable enough to deal with the infinitesimal, he may simply have been mistaken in using a geometrical approach. In addition, Kant, as all person of the modern age, still had a firm and unyielding affirmation that Newton's laws were absolute so perhaps it is unfair to judge this way. The point is, even as the continuity concept becomes almost firmed up by Cauchy, mathematicians and philosophers alike are still unable "to get it."

As a mathematician, I cannot avoid to think of this mathematically but will try. What is the smallest thing and what do it mean to go from one of these small things to another? Without a doubt, this question reeks of the process to go from the infinitesimal to the notion of continuity. What ever claim you make for either, to avoid Kant's solution that both are transcendental knowledge beyond human conception, you must admit there is a process.

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[31] Ibid., Boyer, p. 499-51

Continuity is the process of “going” from one thing to another without skipping any thing along the way. We now know the physics is not as simple as we might have thought. Even if we could answer the question “Is there a smallest part”, the probability of existence at the level of quantum physicals makes the question meaningless without a serious mathematical statement. As Cauchy demonstrates in 1823, a point to point reference in terms of infinitesimals is not a physical property, certainly not in the sense of Newton’s physics.

Conclusion

Perhaps it is not fair to challenge philosophers in the past with what they did not know, and this is partially a result of this paper. But, this is not what was attempted here. This historical tour was meant to show that is was their past, the past knowledge of philosophers of the modern age, that formed a cognitive obstacle to going beyond Zeno’s paradox and problems of continuity and infinity. And yet, perhaps this problem was not last limited to Kant. Not likely. Greek traditions influenced the modern period, and so it would be understandable to expect that modern thinking is having some influence on the contemporary period of thinking. This would be an interesting follow-up.